Categories and Topology

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1 Introduction

1.1 What is the purpose of this lecture?

I decided to make this lecture after realizing that many basic constructions in topology can be explained via commutative diagrams. Category theory is also the natural setting to understand the basic ideas behind algebraic topology. Indeed, homotopy and homology groups are best understood as functors between categories. Category theory actually originated in algebraic topology with the works of S. MacLane and S. Eilenberg. From this point onwards, categories have become useful in many other areas of mathematics and science, from algebraic geometry to mathematical physics and theoretical computer science.

I do not assume anything more than good familiarity with set theory (including disjoint unions and equivalence relations) and some basic understanding of real analysis. Although I sometimes use examples from other areas of mathematics, they are certainly not essential and are just there to accommodate everyone's background. A course on point-set topology can be useful to understand the constructions here since they're the same, except for the fact that the viewpoint used here might be slightly different depending on your course. This lecture is not intended to replace a course on topology, it is meant to be concise and I only develop the notions that I need in order to construct the topologies. Many easy results are left as exercises. You're free to skip them if you want to. The section on the weak-* topology is slightly different from the rest of the material and probably needs some understanding of linear algebra or functional analysis to be fully appreciated.

1.2 What is topology?

A tenet of modern mathematics is that morphisms between mathematical structures are even more useful than the underlying structures. In set theory, we see that bijections preserve the set structure. If A, B are two sets and we have a bijection $\varphi : A \to B$, then A and B are 'essentially the same'. What this means is that you can re-label every element of A with elements of B and not lose any information. In other words, you do not lose any element in the process (surjection) and two distinct elements in B were distinct in A as well (injection). A bijection thus preserves the set structure, but if our set has more structure on top of it, it is certainly not guaranteed that our bijection preserves that structure as well. In full generality, a map that fully preserves a mathematical structure is called an *isomorphism*. An isomorphism between vector spaces preserves the linear structure. An isomorphism between groups preserves the group structure.

The question of topology is then, what do topological isomorphisms preserve? What kind of structures do topological spaces hold? To put it simply, topology is concerned with continuity, and topological isomorphisms (that we will henceforth call *homeomorphisms*) are required to be continuous. But of course, topological spaces are not restricted to subsets of $\mathbb{R}$, which means that continuity on those spaces will have a more general meaning than the usual $\varepsilon - \delta$ definition of continuity. Defining a topology on a space will give a precise meaning to basic intuitions about convergence, continuity, connectedness and compactness which form the crux of elementary real analysis. Topology can be seen as an extension of geometry and studies both local and global properties.

2 A few definitions

2.1 Topology

2.1.1 Topology, Continuity, Convergence

After setting the stage for what we will be concerned in this lecture, we now turn to the basic definition of a topology on a set. Throughout this section, X stands for an arbitrary set.

DEFINITION 2.1.1. A *topology* on a set X is a collection $\mathcal{O}$ of subsets of X such that:

1. X and $\varnothing$ are both in $\mathcal{O}$
2. If $\{A_i\}_{i\in I}$ is a possibly infinite family of elements of $\mathcal{O}$, then their union is in $\mathcal{O}$.
3. If $\{A_i\}_{i\in I}$ is a finite family of elements of $\mathcal{O}$, then their intersection is in $\mathcal{O}$.

Elements of $\mathcal{O}$ are called *open sets*. The pair $(X,\mathcal{O})$ is called a *topological space*.

In other words, a topology on X is a collection of subsets of X that are stable under arbitrary union, finite intersection and contain both the empty set and X itself. Two somewhat natural topologies on any set X usually spring in mind when one first tries to come up examples of topologies.

EXAMPLE 2.1.2. The collection of subsets of X defined by $\mathcal{O} = \{X,\varnothing\}$ is a topology. We usually call it the *trivial* topology because it doesn't hold much information (there are only two open sets). Similarly, the power set of X is also a topology since every subset of X is included. We call this topology the *discrete* topology. Singletons themselves are open in this topology, so there is an enormous amount of open sets. Both topologies are usually not very useful, because in one case the topology is too thin and doesn't hold much information, in the other case the topology usually holds too much information.

DEFINITION 2.1.3. If we have two topologies $\mathcal{O}_1$ and $\mathcal{O}_2$, $\mathcal{O}_1$ is said to be *finer* than $\mathcal{O}_2$ if $\mathcal{O}_2 \subseteq \mathcal{O}_1$. In this case, $\mathcal{O}_2$ is said to be *coarser* than $\mathcal{O}_1$.

These examples, while simple, won't interest us as much as other topologies. We have now defined the notion of a topological space, but it is unclear why it is related to convergence and continuity. Naturally, our definitions of continuity here will need to coincide with the definitions in $\mathbb{R}$. But first, we need to find a topology on $\mathbb{R}$, because we haven't yet shown anything about $\mathbb{R}$. Unsurprisingly, the open sets correspond exactly to the open sets we see in analysis.

DEFINITION 2.1.4. The standard topology on $\mathbb{R}$ denoted $\mathcal{O}_{\mathbb{R}}$ is precisely the set of open sets in $\mathbb{R}$. A set is open in $\mathbb{R}$ precisely if it is of the form $\bigcup_{i=1} U_i$ where U_i are open intervals.

EXERCISE 2.1.5. Show that $\mathscr{O}_{\mathbb{R}}$ is a topology by checking directly the axioms.

In a sense, we can see that the open intervals of $\mathbb{R}$ form a basis for the topology on $\mathbb{R}$. While we won't need this construction, it is a very useful way to define a topology. First you define a basis of sets with certain properties, and then you say the topology generated by that basis is the set of all countable unions of those basis elements.

We now proceeds to define what we mean by converging and continuity. To this end, we need to understand how convergence in $\mathbb{R}$ works. For a sequence to converge in $\mathbb{R}$, we require it to get closer and closer to our limit after a certain index N. While we do not have a general definition of distance in topological spaces, we do have a notion of 'closeness'. X now stands for a topological space $(X, \mathscr{O}_X)$.

DEFINITION 2.1.6. Let $p \in X$ be a point in our topological space. We say V is a *neighborhood* of p if there exists an open set $U \subseteq V$ such that $p \in U$.

The name neighborhood is useful here in visualizing the situation. X itself is a neighborhood of every point $p \in X$ (why?). In $\mathbb{R}$, we can find neighborhoods of points rather easily. Set $\varepsilon > 0$. Then the set $(p-\varepsilon, p+\varepsilon)$ is a neighborhood of p (why?). Another way to say that a sequence $(x_i)_{i=1}^{\infty}$ converges to p is to say that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in (p-\varepsilon, p+\varepsilon)$.

EXERCISE 2.1.7. Show that this definition is equivalent to the usual definition of convergence of a sequence in real analysis.

This viewpoint will be much more useful to define convergence in general topological spaces.

DEFINITION 2.1.8. Let $(x_i)_{i=1}^{\infty}$ be a sequence of elements in X. We say that the sequence *converges* to x if for every neighborhood V of x, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in V$. We write this as $\lim_{n\to\infty} x_n = x$.

To get a sense of how convergence in general topological spaces differ from convergence in $\mathbb{R}$, we will now test these definitions in the case of the discrete and trivial topology.

EXAMPLE 2.1.9 (Convergence in the discrete topology). Let X be an arbitrary set and equip X with the discrete topology. Since every subset of X is open, the singletons themselves are open. Let p be a point in X. Since $\{p\}$ is open, $\{p\}$ is a neighborhood of p. This is a very stringent condition on the convergence of our sequence, because it means every sequence converging to p must be the constant sequence p after a certain N. Indeed, the only way for the sequence to be in $\{p\}$ for all $n \geq N$ is for x_n to be equal to p.

EXAMPLE 2.1.10 (Convergence in the trivial topology). Again, let X be an arbitrary set and equip X with the trivial topology. This time, things are even weirder. Since the only open sets in X are X and $\varnothing$, and $p \notin \varnothing$ for every p, then the only neighborhood of p is X. Surely, since $x_i \in X$ for all i, then our sequence is always contained in our neighborhood. Thus every sequence converges in this topology.

Those two examples show two extremes. On one hand, every sequence converges, on the other, only eventually constant sequences converge. The standard topology on $\mathbb{R}$ lies in between these two topologies. We already see a link between the number of open sets and how easily a sequence converges. The more open sets there are, the harder it is for a sequence to converge in the topology.

We now turn to the other important definition in this chapter, that of a continuous function.

DEFINITION 2.1.11. Let X, Y be two topological spaces. A function $f : X \to Y$ is said to be *continuous* if for every open set $V \in \mathcal{O}_Y$, $f^{-1}(V)$ is open in X. In other words:

$$\boxed{f^{-1}(\mathcal{O}_Y) \subseteq \mathcal{O}_X}$$

It is interesting to note that we require the preimage of an open set to be open in the topology of X, but not the image of an open set to be open in Y. If $f : X \to Y$ is such a map, we say f is an *open map*. A homeomorphism is thus a bicontinuous bijection, that is to say, both $f(x)$ and $f^{-1}(x)$ are continuous functions from X to Y. If there is a homeomorphism between two spaces, we say they are *homeomorphic*, and we write $X \simeq Y$. Two homeomorphic spaces are topologically the same, just like two sets in bijection are set-theoretically the same. The homeomorphism preserves the topological structure and we're effectively just re-labeling elements and open sets.

It is not obvious how this topological definition of continuity is related to the usual $\varepsilon - \delta$ definition of continuity. We now prove that they're the same for the standard topology on $\mathbb{R}$. To this end, we start by defining 'open balls'.

DEFINITION 2.1.12. Let $\varepsilon > 0$ be a real number. The set $B_\varepsilon(x) := \{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$ is called the *open ball* of radius ε and center x.

We can now redefine open sets in $\mathbb{R}$ in terms of open balls. A subset A of $\mathbb{R}$ is open if for every $x \in A$, there exists a $\delta > 0$ in $\mathbb{R}$ such that $B_\delta(x) \subseteq A$. We leave the proof of the equivalence as an exercise. You are free to give a proof by authority say it is trivial if you wish to do so. We now turn to our first theorem.

Theorem 2.1.13 (Equivalence of the definitions of continuity in $\mathbb{R}$)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous in the usual sense if and only if it is continuous in the topological sense.*

Proof. We start with the first implication. Suppose that f is continuous. Let V be an open set in $\mathbb{R}$. Let $x = f^{-1}(V)$. Since V is open, there exists an ε such that $B_\varepsilon\left(f(x)\right) \subseteq V$ by our previous characterization of open sets in $\mathbb{R}$. We now use the continuity of f. Since f is continuous, there exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon\left(f(x)\right)$. Therefore we have that $f(B_\delta(x)) \subseteq V$ and thus $B_\delta(x) \subseteq f^{-1}(V)$. This shows $f^{-1}(V)$ is open in $\mathbb{R}$ because of our earlier characterization, again.

Suppose now that f is continuous in the topological sense. Let $x \in \mathbb{R}$. We know that for $\varepsilon > 0$, $B_\varepsilon(f(x))$ is open in $\mathbb{R}$ (why?). Since f is continuous, $f^{-1}\left(B_\varepsilon(f(x))\right)$ is open. Since $f(x) \in B_\varepsilon(f(x))$, we have that $x \in f^{-1}\left(B_\varepsilon(f(x))\right)$. Again, since f is continuous, there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}\left(B_\varepsilon(f(x))\right)$. This shows continuity in the $\varepsilon - \delta$ sense (why?). □

Once again, let us contrast continuity in $\mathbb{R}$ with our two examples from earlier, the discrete and trivial topology.

EXAMPLE 2.1.14. Let X be an arbitrary set. Equip X with the discrete topology. Then every function $f : X \to Y$ where Y is an arbitrary topological space is continuous. Indeed, $f^{-1}(V) \subseteq \mathscr{O}_X$ for all $V \in \mathscr{O}_Y$, since every subset of X is in $\mathscr{O}_X$.

We leave the other example as an exercise.

EXERCISE 2.1.15. Characterize all the continuous functions $f : X \to Y$, where X is an arbitrary topological space and Y is an arbitrary set equipped with the trivial topology $\mathscr{O}_Y = \{Y, \varnothing\}$.

EXERCISE 2.1.16. Show that the composition of continuous functions between topological spaces is still continuous.

2.1.2 Connectedness and Path-Connectedness

An important question is that of invariants. A *topological invariant* is an abstract property preserved via homeomorphisms. What this means is that if $X \simeq Y$, then X and Y both possess the same invariants. A well-known example of an invariant (that happens to be topological) is the Euler number of a polyhedron. The real usefulness of this statement lies in its contrapositive. Let ψ be a topological invariant that computes a certain number. For instance, $\psi(X) = 2$. If $\psi(X) \neq \psi(Y)$, X cannot be homeomorphic to Y. We will concern ourselves with three invariants in this lecture: connectedness, path-connectedness and compactness. In this section, we will define these properties and show they are topological invariants. We begin with (path-)connectedness.

An interesting property of $\mathbb{R}$ is that intervals are 'connected'. There is no hole between two elements of (a, b), unlike in $\mathbb{Q}$ where the irrationals puncture any 'interval in $\mathbb{Q}$'. For instance, if we consider $(0,2) \in \mathbb{R}$, every $0 < x < 2$ is in $(0,2)$, but $\sqrt{2}$ isn't in $\mathbb{Q}$. There is a 'hole'. In fact, there are many such holes if we restrict our interval to $\mathbb{Q}$, because $\mathbb{Q}$ is dense in $\mathbb{R}$. Unlike continuity and convergence, connectedness is expressed the same way in $\mathbb{R}$ and in general topological spaces. There are many equivalent definition of connectedness, but we will restrict ourselves to only one for now.

DEFINITION 2.1.17. A topological space X is said to be *connected* if X cannot be made into a disjoint union of non-empty subsets $A, B \in X$.

In the special case where $X = \mathbb{R}$ with its usual topology, it is very clear that $(0,1) \cup (2,5)$ isn't connected. Intervals are however. We will not prove this fact extensively, because we are more interested in general spaces than $\mathbb{R}$. We do leave part of it as an exercise however.

EXERCISE 2.1.18. Show that the open interval (a, b) is connected in $\mathbb{R}$. Show that $\mathbb{R}$ is connected.

Another topological invariant that is related to connectedness is path-connectedness.

DEFINITION 2.1.19. Let X be an arbitrary topological space. We say X is *path-connected* if for every two points $x, y \in X$, there exists a continuous function $\varphi : [0,1] \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$. Such a continuous function is called a *path* from x to y.

Proposition 2.1.20. *A path-connected topological space is connected.*

Proof. We prove this by contradiction. Suppose that $X = A \cup B$, where $A \neq B \neq \varnothing$ and $A \cap B = \varnothing$. We choose $a \in A$ and $b \in B$. Since X is path-connected, there exists a path φ between a and b. By continuity: $\varphi^{-1}(X) = \varphi^{-1}(A \cup B) = \varphi^{-1}(A) \cup \varphi^{-1}(B) = [0,1]$. But $[0,1]$ is connected, contradiction. □

We now show those properties are invariants.

Theorem 2.1.21. *Connected and path-connectedness are topological invariants.*

Proof. We only prove path-connectedness, connectedness is left as an exercise. The proofs are similar. Let X be a path-connected topological space, and let $f : X \to Y$ be a homeomorphism. Let $a, b \in Y$. Since f is a homeomorphism, there exists $a', b' \in X$ such that $f(a') = a$, $f(b') = b$. Since X is path-connected, there exists a path φ from a' to b'. Since $\varphi(0) = a'$ and $\varphi(1) = b'$, we have that $f \circ \varphi(0) = a$ and $f \circ \varphi(1) = b$, which shows that $f \circ \varphi : [0,1] \to Y$ is a path between a and b in Y. □

EXERCISE 2.1.22. Prove that connectedness is a topological invariant.

2.1.3 Compactness

Compactness is a familiar property to anyone having taken a basic introduction to analysis. The importance of compactness is hard to overstate. Very important results and constructions in real analysis depend crucially on the compactness of the interval $[a, b]$. The Riemann integral is first defined over a compact interval before we consider improper integrals over non-compact sets. Two very important results state that a continuous function defined over an integral possesses a maximum, and that it is uniformly continuous. The proofs of those statements can be found in any books on basic real analysis. While there is no direct generalization of uniform continuity in topological spaces in general (there is one for a particular subset of them), compactness in topological spaces is still important. We shall devote this section to defining compactness in general spaces, prove that it is a topological invariant and finally prove that over $\mathbb{R}$, both definitions imply one another, which is the content of the celebrated Heine-Borel theorem.

We first define what an open cover is.

DEFINITION 2.1.23. We say that a family of open sets $\{A_i\}_{i \in I}$ is an *open cover* of X if $X \subseteq \bigcup_{i=1} A_i$.

Equipped with this notion of covers, we can now state what it means for a topological space to be compact.

DEFINITION 2.1.24. X is a *compact* topological space if, for every open cover $\{A_i\}_{i \in I}$, there exists a finite subcover. That is, for every $\{A_i\}_{i \in I}$, $X \subseteq \bigcup_{i=1}^{n} A_i$. Only finitely many A_i's are required to cover X.

EXAMPLE 2.1.25. Any finite topological space is compact (why?).

It is not immediately obvious that the definition in $\mathbb{R}$ (closed and bounded set) is equivalent to the more general definition. In fact, there are many definitions of compactness. Another one is that a space is compact if every sequence has a convergent subsequence. In the context of $\mathbb{R}$, this is the Bolzano-Weierstrass theorem, but in general spaces, this is called sequential compactness. It is not unusual that great theorems turn into definitions when we go up a notch in abstractness.

Before showing the Hahn-Banach theorem, we demonstrate the invariance of compactness.

Theorem 2.1.26 (Compactness is a topological invariant)**.** *Let X be a compact topological space, and let $\varphi : X \to Y$ be a homeomorphism. Then Y is compact.*

Proof. Let F_Y be an open cover for Y. Then $f^{-1}(F_Y) := \left\{f^{-1}(U) \mid U \in F_Y\right\}$ is an open cover for X. By compactness of X, there exists a finite subcover of $f^{-1}(F_Y)$. But then $F'_Y = \{V_1, \ldots, V_n\}$ is a finite subcover of Y. □

Now that we know compactness is a topological invariant, we can show some interesting things. We leave the details to the reader.

EXERCISE 2.1.27. Show that $\mathbb{R}^n$ is not homeomorphic to $[0,1]^n$. What about $(0,1)^n$?

As we have done for continuity, it is important to check whether our definition of compactness coincides with the usual concept of compactness on $\mathbb{R}$. This important theorem shows that the open cover definition of compactness is an accurate generalization of the concept of being 'closed and bounded'. The proof can be found in any standard real analysis text and isn't particularly illuminating, so we won't mention it here.

Theorem 2.1.28 (Heine-Borel)**.** *A set $K \in \mathbb{R}$ is compact if and only if it is closed and bounded.*

2.2 Categories

2.2.1 Categories and morphisms

As we said in the introduction, categories have become rather ubiquitous in mathematics. We are going to define categories and give a couple of examples. You do not need to understand every single example here to understand the material. Before defining categories, we need to make a small remark on a finer point. As Russell's paradox shows, the notion of a set of all set can lead to some trouble. The famous paradox discusses the set of all sets that are not members of themselves. To avoid any kind of set-theoretical trouble, we will use *classes*. Classes are collections of sets. For instance the class of all sets is well-defined. Do notice that this class is *not* a set!

DEFINITION 2.2.1. A *category C* is defined by:

1. A class Obj(C) of elements of C, called *objects*

2. A class $\hom_C[X,Y]$ of maps between every $X,Y \in \mathrm{Obj}(C)$, called *morphisms*

3. A composition of morphisms $\circ : \hom_C[Y,Z] \times \hom_C[X,Y] \to \hom_C[X,Z]$ such that:

   (a) The composition is associative: for $f : A \to B$, $g : B \to C$, $h : C \to D$, we have $f \circ (g \circ h) = (f \circ g) \circ h$.

   (b) Every element $X \in \mathrm{Obj}(C)$ has an identity morphism $1_X : X \to X$ such that for every $f : A \to B$ we have $1_A \circ f = f = f \circ 1_B$.

This definition can seem quite general. We have some elements and some morphisms between them. It turns out that this is general enough to encompass a large collection of interesting mathematical structures, but specific enough that we can state meaningful theorems about these structures. Let us see some example. We leave it to the reader to show these examples are categories.

EXAMPLE 2.2.2.

1. The category **Set** consisting of the class of all sets along with set functions between every object of the category.

2. The categories **Grp**, **Ring**, **Field** of all groups/rings/fields along with groups/rings/fields homomorphisms.

3. The category **Ab** of all abelian groups with group homomorphisms.

4. The category $\mathbf{Vect_k}$ of all vector spaces over the field k.

5. The category **Top** of topological spaces along with, you guessed it, continuous functions.

6. The category **Set*** of pointed sets (sets with a base point), that is pairs of the form (X,x) with $x \in X$ and morphisms that take base points to base points.

7. The category **Top*** of pointed topological spaces, with continuous functions that take base points to base points.

Notice how all these examples consist of a set along with some additional structure and/or a base point, and morphisms that preserve the structure between objects of the category. Categories which are built upon **Set** (like **Grp** or **Top**) with additional structure are called *concrete categories.* Notice how **Ab** and **Grp** have the same kind of morphisms. In a sense to be defined, **Ab** is a 'subcategory' of **Grp**, since **Ab** consists of those group which also happen to be abelian. We will see later on how we can define more precisely this idea of restricting structure on a category.

We saw that different mathematical structures have different isomorphisms between them. Isomorphism can be defined in general for categories as well.

DEFINITION 2.2.3. Let C be a category and let $X,Y \in \mathrm{Obj}(C)$. We say that a morphism $f : X \to Y$ is an *isomorphism* if there exists a morphism $g : Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. An *endomorphism* is a morphism $f : X \to X$. An endomorphism that happens to be an isomorphism is called an *automorphism.*

EXAMPLE 2.2.4. A homeomorphism is an isomorphism in the category **Top**. A bijection is an isomorphism in the category **Set**.

We have defined morphisms between objects of a category. It seems rather natural to ask whether there are functions between categories. Do notice that such functions should be defined both for objects and morphisms of categories. Looking back on our example with **Ab** and **Grp**, surely there should be a way to say that we ignore the 'commutativity' of abelian groups to get back our group without the abelian structure. And there is in fact a way to do this.

2.2.2 Functors and diagrams

DEFINITION 2.2.5. Let C_1, C_2 be two categories. A *functor* $\mathscr{F} : C_1 \to C_2$ is a map that assigns to each object $X \in C_1$ an object $\mathscr{F}(X) \in C_2$ and to each morphism $f : X \to Y$ in C_1 a morphism $\mathscr{F}(f) : \mathscr{F}(X) \to \mathscr{F}(Y)$ such that:

1. $\mathscr{F}(1_X) = 1_{\mathscr{F}(X)}$ (respects identities),
2. $\mathscr{F}(f \circ g) = \mathscr{F}(f) \circ \mathscr{F}(g)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in C_1 (respects composition).

We can now make clearer the idea of forgetting structure.

EXAMPLE 2.2.6. The group *forgetful functor* $\mathscr{F}$: **Grp** → **Set** is the functor that sends a group to its underlying set and homomorphisms to their underlying set function. It 'forgets' the group structure. While not very formally defined, this is a common example of a simple functor. A similar example can be given from **Top** to **Set** where one forgets the topological structure.

In the last part of this lecture, we will see why category theory had its origins in algebraic topology. Indeed, assigning an algebraic invariant to a topological space is similar to defining a functor from **Top** to another (algebraic) category. Functors in general can be used to describe a wide range of mathematical entities, from something as simple as a topological invariant to certain quantum field theories in physics. Before moving on to constructing new topologies, we need a few more tools.

DEFINITION 2.2.7. We define the *opposite category* C^{op} of a category C to be the category formed with the same objects of C, but such that the morphisms are reversed, i.e. we associate to any two objects $X, Y \in C^{\text{op}}$ the class $\hom_C[Y, X]$, that is, maps of the form $f : Y \to X$ and we fix $\text{Obj}(C^{\text{op}}) = \text{Obj}(C)$.

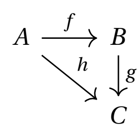
DEFINITION 2.2.8. A functor is said to be *contravariant* if it is of the form $\mathscr{F} : C^{\text{op}} \to D$ for some categories C, D. It is similar to a normal functor (called a *covariant* functor) except that it reverses the morphisms and the composition.

EXAMPLE 2.2.9. The functor $* : \mathbf{Vect_k} \to \mathbf{Vect_k}$ sends a vector space to its algebraic dual space. Let V be a k-vector space. We send V to $V^* := \{\varphi : V \to k \mid \varphi \text{ is linear}\}$ and we send each linear map $f : V \to W$ to its dual map $f^* : W^* \to V^*, \alpha \mapsto \alpha \circ f$. Furthermore, we know that $1_V^* = 1_{V^*}$ and that $(f \circ g)^* = g^* \circ f^*$ which proves that $*$ is a contravariant functor.

The main goal of this lecture is to define common topologies using commutative diagrams. We are now equipped with the basic definitions of category theory and topology required to make these constructions. We end this section by defining diagrams and what it means for them to commute.

DEFINITION 2.2.10. A *diagram* is a collection of objects of a certain category and maps (usually called *arrows*) between them. We say the diagram commutes if every direct path along the arrows lead to the same result.

EXAMPLE 2.2.11. The following simple diagram commutes if $g \circ f = h$:



Commutative diagrams are really useful tools in category theory and homological algebra. They are kind of similar to equalities in algebra, as in they define important properties of categories. For instance, many categories are defined as categories whose objects or morphisms satisfy certain commutative diagrams. There is a more formal way to define diagrams, namely as a functor from an index category to another category. We will not use this definition here because we want to focus on topology and the intuitive definition works better for our purpose.

3 Topological constructions

We now come to the main part of the lecture. Equipped with category theory and more specifically commutative diagrams, we define the basic topological constructions that one can find in any elementary textbook on topology. Usually, these notions will be defined differently. For instance, the subspace topology is defined with intersections of open sets. I strongly believe that the 'categorical' definition (with a commutative diagram) is clearer in that it shows what's happening behind the scenes. It is not clear what the product and subspace topology have in common, but once they are defined in terms of diagrams, the connection is immediately obvious. We begin by defining two very important general constructions that are dual to one another.

3.1 Final and Initial topology

The idea behind the initial topology is to 'pull back' the topology on a family of spaces to an arbitrary set, thereby constructing a topological space $(X, \mathcal{O})$. To this effect, we will use a family of functions.

DEFINITION 3.1.1 (Initial topology). Let X be an arbitrary set, let $\{Y_i\}_{i \in I}$ be an indexed family of topological spaces and let $\{f_i\}_{i \in I}$ be a family of function $f_i : X \to Y_i$. The *initial topology* $\mathcal{O}_I$ on X is the coarsest topology on X such that each function f_i is continuous.

While this definition is rather general and abstract, it is exactly what we need to precisely define some important examples of topologies. Before doing that, we introduce its dual notion, the final topology. Notice the 'arrow reversal' that is characteristic of a dual construction.

DEFINITION 3.1.2 (Final topology). Let X be an arbitrary set, let $\{Y_i\}_{i\in I}$ be an indexed family of topological spaces and let $\{f_i\}_{i\in I}$ be a family of function $f_i : Y_i \to X$. The *final topology* on X is the finest topology such that each function f_i is continuous.

The difference in these two definitions is the domains/codomains of the functions. This time, the idea behind the final topology is to 'push forward' the topology on X via the topology of each Y_i and of our family of functions. We now use commutative diagrams to describe how topological spaces equipped with initial/final topology behave with respect to continuous maps from/to other morphisms.

Theorem 3.1.3 (Characteristic property of the initial topology)**.** *Let X be a topological space with the initial topology and let Z be a topological space. The function $\varphi : Z \to X$ is continuous if and only if $f_i \circ \varphi$ is continuous for all $i \in I$.*

$$\begin{array}{ccc} X & \xrightarrow{f_i} & Y_i \\ {\scriptstyle\varphi}\uparrow & \nearrow_{f_i\circ\varphi} & \\ Z & & \end{array}$$

Likewise, we have a characteristic property for the final topology.

Theorem 3.1.4 (Characteristic property of the final topology)**.** *Let X be a topological space with the initial topology and let Z be a topological space. The function $\varphi : X \to Z$ is continuous if and only if $\varphi \circ f_i$ is continuous for all $i \in I$.*

$$\begin{array}{ccc} Y_i & \xrightarrow{f_i} & X \\ & {}_{\varphi\circ f_i}\searrow & \downarrow{\scriptstyle\varphi} \\ & & Z \end{array}$$

These properties will naturally transfer over to the topologies we will construct in the next section.

3.2 Products

3.2.1 First attempt: The Box Topology

We finally begin constructing various examples of important topologies. The first is that of a Cartesian product of topological spaces. The most intuitive case is probably that of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. We have already defined a topology on $\mathbb{R}$. We wish to extend this topology to $\mathbb{R}^2$. A natural generalization is to take Cartesian products of open sets in $\mathbb{R}$. Namely, we set $\mathscr{O}_{\mathbb{R}^2} = \mathscr{O}_{\mathbb{R}} \times \mathscr{O}_{\mathbb{R}} := \{U \times V \mid U, V \in \mathscr{O}_{\mathbb{R}}\}$.

EXERCISE 3.2.1. Prove that $\mathbb{R}^2$ equipped with the topology above is a topological space.

While simple, this generalization works well for $\mathbb{R}^2$. In fact, it works well for any n by using a similar argument that you gave in the exercise. But there is nothing special about the topology on $\mathbb{R}$ with respect to this product, indeed we have never used any properties specific to $\mathbb{R}$ to define this topology. This suggests a more generalized definition of this topology.

DEFINITION 3.2.2 (The Box Topology). Let $(X_i, \mathscr{O}_i)$ be a family of topological spaces. Let $X = \prod_{i=1} X_i$ denote the Cartesian product of our topological spaces. We can define a topology on this product called the *box topology* by setting $\mathscr{O}_X := \{\prod_{i=1} U_i \mid U_i \in \mathscr{O}_i\}$.

It is not difficult to prove that the box topology is, in fact, a topology. As you may have guessed from the title of this subsection, this isn't the only possible topology we can put on our product. When we take finite products, this topology is totally reasonable and our intuition is still correct. The problems start occurring when we take infinite products.

DEFINITION 3.2.3. We define the set of real sequences $\mathbb{R}^\omega$ to be $\mathbb{R}^\omega := \prod_{i=1}^\infty \mathbb{R}_i$.

If we equip $\mathbb{R}^\omega$ with the box topology, we soon get some very concerning problems. The box topology contains many open sets, too many for basic results to hold true. The following simple example shows that continuity in the components does not mean continuity of the function as a whole.

EXAMPLE 3.2.4. Let $\mathbb{R}^\omega$ be the space of real sequences with the box topology. Consider the function $f : \mathbb{R} \to \mathbb{R}^\omega$ that is the identity in every component: $f : x \mapsto (x, x, x, \dots)$. It is clear that the component functions $f_i : x \mapsto x$ are continuous because the identity is continuous on $\mathbb{R}$. Yet this function is *not* continuous. Suppose f is continuous. We consider the open set

$$U = \prod_{i=1}^{\infty} \left(-\frac{1}{i}, \frac{1}{i}\right).$$

$f(0) = (0, 0, 0, \dots) \in U$. By continuity of f, there should exist a small neighborhood $(-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset f^{-1}(U)$. But this implies that

$$f\left(\frac{\varepsilon}{2}\right) = \left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \dots\right) \subset U$$

which is clearly false since $\frac{\varepsilon}{2} > \frac{1}{n}$ for $n > \frac{2}{\varepsilon}$.

This little example shows just how bad our topology really is for infinite products. The problem is that there are way too many open sets. We need a coarser topology. This example should underlie why initial and final topology are important. They are the coarsest/finest topology such that continuity can be described via component functions, unlike the box topology we have constructed in this section. This begs for an alternate definition of a topology on a product.

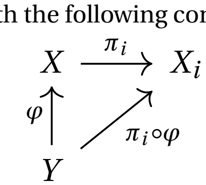
3.2.2 Second attempt: The Product Topology

This time around, we will construct a product topology on $\prod_{i=1}^{\infty} X_i$ via an initial topology.

DEFINITION 3.2.5 (Canonical projections). Let $X = \prod_{i=1}^{\infty} X_i$. We define the *canonical projections* to be functions $\pi_i : X \to X_i$ that send an element $x = (x_1, x_2, \ldots, x_i, \ldots)$ to the i-th coordinate x_i.

The canonical projections define a family of functions indexed by I, associated to a family of topological spaces X_i. We now use these projections to define our topology.

DEFINITION 3.2.6. The *product topology* is the initial topology with respect to the canonical projections. That is, it is the coarsest topology on X such that each π_i is continuous. We can describe it with the following commutative diagram:



φ is continuous if and only if its composition with each component is continuous.

This topology is much better behaved than the box topology. In the finite case, they coincide. The reader is free to try to prove that the function defined in 3.2.4 is now continuous in this topology. The open sets in a space equipped with the product topology are of the form $U = \{\prod_{i=1} U_i \mid U_i \in X_i \text{ and } U_i \neq X_i \text{ for only finitely many } i\}$.

3.3 Subspaces

The case of subspaces is easier to handle. There are actually two equivalent ways to define the subspace topology. The first one is intuitive and the usual definition in most standard introductions to topology.

DEFINITION 3.3.1 (Subspace Topology: I). Let $(X, \mathcal{O}_X)$ be a topological space, and let A be a subset of X. The *subspace topology* on A is defined as $\mathcal{O}_A := \{U \cap A \mid U \in \mathcal{O}_X\}$.

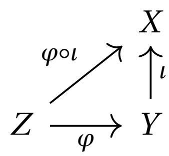
EXERCISE 3.3.2. Show this is a topology.

What is sometimes not known to students new to topology is that this definition is precisely the initial topology with respect to the inclusion map.

DEFINITION 3.3.3. The *inclusion map* is the map $\iota : A \to X$ where $A \subseteq X$ that sends A to itself in X.

We now give our definition of the subspace topology.

DEFINITION 3.3.4 (Subspace Topology: II). The subspace topology is the coarsest topology on A such that the inclusion map is continuous. We can describe it with the following diagram:



Thus a function to the subspace $A \subseteq X$ is continuous if and only if it is continuous in X when composed with the inclusion map.

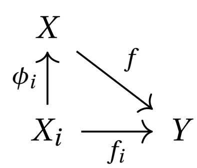
3.4 Dual of the product topology: disjoint unions

We want to equip disjoint unions of topological spaces with a topology. This time, we will use a final topology to do this.

DEFINITION 3.4.1. Let $X = \bigsqcup_{i=1} X_i$ be the disjoint union of the indexed family $\{X_i\}_{i \in I}$. We define the *canonical injections* $\phi_i : X_i \to X$ defined by $\phi(x) = (x, i)$.

We will use these injections to define our topology.

DEFINITION 3.4.2. Let $X = \bigsqcup_{i=1} X_i$ be the disjoint union of the indexed family $\{X_i\}_{i \in I}$. The *disjoint union topology* is the finest topology on X such that the canonical injections stay continuous. This topology can be characterized by the following universal property: If Y is a topological space, and $f_i : X_i \to Y$ is a continuous function for each $i \in I$, then there exists a unique continuous map $f : X \to Y$ such that the following diagram commutes:



3.5 Dual of the subspace topology: quotient spaces

The intuition behind quotient spaces is altogether more involved than for subspaces, or product. The geometric idea is that we want to identify (or glue) certain points together, to form a smaller space. To do that, we will define an equivalence relation on our space, and then take our space to be the equivalence classes (which necessarily partition the entire space). This is in fact similar to what we do with groups when we define quotient groups. The equivalence relation was simply hidden in the cosets. Remember from set theory that if $x \in [y]$ and $x \in [z]$, then $[y] = [z]$.

DEFINITION 3.5.1. Let X be a topological space and let $\sim$ be an equivalence relation on X. We define the *quotient space* $X/\sim$ to be the set $X/\sim := \{[x] \mid x \in X\}$.

Our goal in this section is to define a topology on $X/\sim$. To this extent, we will use a special map and define the topology on $X/\sim$ to be the final topology with respect to this map.

DEFINITION 3.5.2. Let $X/\sim$ be a quotient space. We define the *canonical quotient map* to be $f : X \to X/\sim$, $x \mapsto [x]$.

DEFINITION 3.5.3. Let $X/\sim$ be a quotient space. The *quotient topology* on $X/\sim$ is the final topology with respect to the canonical quotient map. The quotient space and the canonical quotient map are characterized by the following universal property: Let g be a continuous function from $X \to Y$ such that $x \sim y$ implies $g(x) = g(y)$ for all $x, y \in X$. Then, there exists a unique continuous map $f : X/\sim \to Y$ such that $g = f \circ q$. We say that g descends to the quotient.

EXAMPLE 3.5.4. Let D^2 denote the two-dimensional disk $D^2 := \{x \in \mathbb{R}^2 \mid \|x\| \le 1\}$. Denote its boundary by ∂D^2. Then $D^2/\partial D^2 \simeq S^2$.

EXAMPLE 3.5.5. Let $S^1 := \{x \in \mathbb{C} \mid \|x\| = 1\}$ denote the unit circle. We have that $\mathbb{R}/\mathbb{Z} \simeq S^1$ by the homeomorphism $x \mapsto e^{2\pi i x}$.

3.6 Weak-* topology

DEFINITION 3.6.1. Let K be a field equipped with a topology. We say that V is a *normed vector space* if it is a vector space over K equipped with a norm. The topology induced by the norm will be called the *strong topology* on V.

From now on, K stands for either $\mathbb{R}$ or $\mathbb{C}$.

EXAMPLE 3.6.2. We can equip $\mathbb{R}^n$ with the usual Euclidean norm. It is clear that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Our goal in this section will be to define another topology on V that will be useful in functional analysis.

DEFINITION 3.6.3. Let V be a vector space over K. The *continuous dual space* of V is the vector space of bounded linear functionals $V^* := \{\varphi : V \to K \mid \varphi \text{ is linear and bounded}\}$. The *double dual space* is then defined as $V^{**} := \{\varphi : V^* \to K \mid \varphi \text{ is linear and bounded}\}$.

We now recall from functional analysis that there is an injection into the double dual space (an isomorphism if V is finite-dimensional). We start by constructing the evaluation map that takes a linear functional and applies it to elements of V.

DEFINITION 3.6.4. Let V be a normed vector space. The *evaluation map* $\mathrm{ev}_v : V^* \to K$ is the linear map $\varphi \mapsto \varphi(v)$. We then define the *double-dual injection* to be the map $\Psi_v : V \to V^{**}$ such that $v \mapsto \mathrm{ev}_v$, for all $v \in V$.

It is with respect to these evaluation maps that we are going to define the weak-* topology on V^*.

DEFINITION 3.6.5 (Weak-* topology). The *weak-* topology* on V^* is the coarsest topology on V^* such that every map ev_v stays continuous. In other words, it is the initial topology with respect to the maps ev_v. This topology is useful for multiple reasons, one of which is the celebrated Banach-Alaoglu theorem from functional analysis which states that the closed unit ball of V^* is compact in the weak-* topology.