Topics in Foreign Exchange Markets

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1 Motivation

1.1 Historical Overview

Modern foreign exchange began in 1880—the year when the gold standard began. The gold standard was a monetary system where paper money was defined and exchanged at fixed quantities of pure gold. Exchange rates were generally allowed to fluctuate $\pm 1\%$. Since the end of the gold standard in 1971, however, exchange rates have been mostly determined by market forces. Some countries decide to "peg" their currency, to fix the exchange rate to a certain amount of another currency. The fluctuation of exchange rates (or the stability of a peg) presents a significant risk to foreign investment, overseas revenue, and trade.

1.2 Target Audience and Objective

Intended for mathematics undergraduates with little finance knowledge, this lecture provides a detailed introduction to the largest market in the world.

2 Market Structure and Derivatives

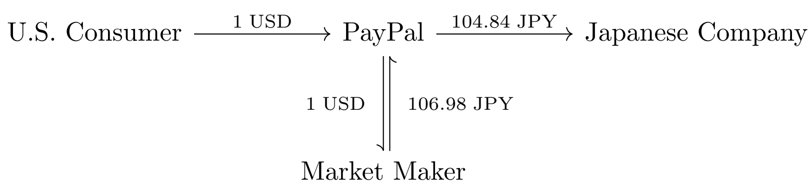
2.1 Microstructure of Foreign Exchange

Exchange rates are quoted as a *currency pair* X/Y, where X is the *base currency* and Y is the *quote currency*. For instance, in USD/JPY, the base currency is the U.S. dollar (USD) whereas the quote currency is the Japanese yen (JPY). The value of this currency pair denotes a quote for U.S. dollars in Japanese yen. Below is a table of some popular currency pairs.

Currency Pair	Description	Value
EUR/USD	Euro/U.S. Dollar	1.10 USD
USD/JPY	U.S. Dollar/Japanese yen	106.98 JPY
GBP/USD	British Pound/U.S. Dollar	1.23 USD
USD/CHF	U.S. Dollar/Swiss Franc	0.99 CHF

Table 1: Exchange rates as of 2019 September 7 at 18:15 JST

To buy currency, one needs a seller, and vice versa. There is often someone called a *market maker*, who takes the other side of the buyer or seller's transaction. Market makers are incentivized to "make a market" by collecting the spread between the price they buy currency—the *bid price*—and the price they sell currency—the *asking price*. In short, market makers profit from facilitating trade. The majority of market makers are banks and trading firms. Foreign exchange is mostly over-the-counter, so exchange rates may differ across market makers. In practice, they are almost identical due to arbitrage, but there are exceptions. PayPal, for instance, quotes dollar-yen rates about 2% lower than banks since arbitrage through PayPal is impossible. This makes foreign exchange a highly profitable business for PayPal.



The best exchange rate is not ensured when dealing with individual market makers, as shown above. Instead of market makers, one can use an *electronic communications network* (ECN): a network where quotes are constantly posted by market participants and orders are matched. The main advantage of ECNs is that they offer the best bid/ask price possible, given the quotes from all market participants connected to the network. ECNs profit from commissions on trades. There are two types of ECNs: retail and institutional. The former is for individuals whereas the latter is for large corporations and investment companies. In either case, ECNs provide a quick way to trade currency.

Prices are ultimately set by the chaotic actions of market participants. While economic sentiment influences rates in the long-term, the short-term is random.

2.2 A Model for Exchange Rates

Suppose there are riskless American and Japanese assets with returns r_1 and r_2, respectively. Then we can model these assets as follows.

$$dA = r_1 A(t) \tag{2.2.1}$$

$$dB = r_2 B(t) \tag{2.2.2}$$

Notice this setup as is normalizes the initial values of $A(t)$ and $B(t)$ to 1. Let the dollar-yen exchange rate follow geometric Brownian motion (GBM), that is, satisfy the stochastic differential equation

$$dY_t = \mu Y_t \, dt + \sigma Y_t \, dW_t \tag{2.2.3}$$

where μ is the *percentage drift* and σ is the *percentage volatility*. We use GBM to avoid negative values and model relative changes in Y_t as Brownian motion. To solve equation (2.2.3), we will use Itô calculus, but first define some things.

Definition 2.2.1. Let $W(t)$ be a Wiener process (Brownian motion) with filtration $\mathcal{F}_t$, and let $0 = t_0 < \cdots < t_n = t$ be a partition P. For a locally bound, left-continuous and $\mathcal{F}_t$-adapted process $X(t)$, we define the *Itô integral*

$$\int_0^t X(s) \, dW_s = \lim_{||P|| \to 0} \sum_{k=0}^{n-1} X(t_k)(W(t_{k+1}) - W(t_k))$$

where $||P||$ is the mesh of the partition P.

Definition 2.2.2. An adapted process $X(t)$ is an *Itô process* if it can be written as the sum of an integral with respect to time and an Itô integral.

Theorem 2.2.3. *Let $dX_t = \mu(t,X)\,dt + \sigma(t,X)\,dW_t$ be an Itô process and $f(t,X(t))$ be a twice-differentiable scalar function. Then*

$$\begin{aligned} df &= f'(t,X(t))\,dX_t + \frac{1}{2} f''(t,X(t))\,(dX_t)^2 \\ &= \left(\frac{\partial f}{\partial t} + \mu(t,X)\frac{\partial f}{\partial X} + \frac{1}{2}\sigma(t,X)^2 \frac{\partial^2 f}{\partial X^2} \right) dt + \sigma(t,X)\frac{\partial f}{\partial X}\,dW_t. \end{aligned}$$

We are now ready to solve equation (2.2.3). Apply Itô's lemma (Theorem 2.2.3) with $f(Y(t)) = \ln(Y(t))$ to obtain

$$df = \left(\frac{\partial f}{\partial t} + \mu Y \frac{\partial f}{\partial Y} + \frac{1}{2}\sigma^2 Y^2 \frac{\partial^2 f}{\partial Y^2} \right) dt + \sigma Y \frac{\partial f}{\partial Y}\,dW_t \tag{2.2.4}$$

$$= \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma\,dW_t. \tag{2.2.5}$$

Integrating both sides and simplifying, we obtain

$$Y(t) = Y(0) \exp\left(\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W(t) \right). \tag{2.2.6}$$

The value of the riskless American asset in Japanese yen is then

$$A(t)Y(t) = Y(0) \exp\left(\left(\mu + r_1 - \frac{1}{2}\sigma^2 \right) t + \sigma W(t) \right). \tag{2.2.7}$$

In order to price derivatives, it is often useful to look at asset prices under an *equivalent martingale measure* such as the following.

Definition 2.2.4. Let $X(t)$ be a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A probability measure $\mathbb{Q}$ is a *risk-neutral measure* if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $X(t)$ is a martingale under this measure.

The motivation behind the risk-neutral measure is to be able to discount expected values of assets with only one value—the risk-free interest rate—as opposed to several values. But what do we mean by "discount?"

Definition 2.2.5. Suppose $r(t)$ is an interest rate process. Then the process

$$D(t) = \exp\left(-\int_0^t r(s)\,ds\right)$$

is called a *discount process.*

Note that $r(t)$ is either random or deterministic, depending on the time horizon. Over short periods of time, our return is certain. We are uncertain over long periods of time due to not knowing the future interest rate. The motivation behind this definition is to determine the *present value* of an investment, since we need consistent times and dates in order to compare payoffs from different investments.

Let $\mathbb{Q}_1$ and $\mathbb{Q}_2$ represent the risk-neutral measure for the American and Japanese investor, respectively. To compute the exchange rate (equation (2.2.3)) under $\mathbb{Q}_2$, we require the process

$$\exp(-r_2 t)A(t)Y(t) = \exp\left(\left(\mu + r_1 - r_2 - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right) \tag{2.2.8}$$

to be a martingale. Since $\exp(-\frac{1}{2}\sigma^2 t + \sigma W(t))$ is already a martingale, we only need the condition $\mu = r_2 - r_1$. Now let's consider the exchange rate under $\mathbb{Q}_1$. This time we have

$$\exp(-r_1 t)\frac{B(t)}{Y(t)} = \exp\left(\left(r_2 - r_1 - \mu + \frac{1}{2}\sigma^2\right)t - \sigma W(t)\right) \tag{2.2.9}$$

which is a martingale when $\mu = r_2 - r_1 + \sigma^2$. Unless the risk-free interest rate in the U.S. and Japan are identical, both measures necessarily disagree on the drift coefficient! This paradoxical outcome reminds to rely on risk-free measures not as a price process, but as a tool for arbitrage-free pricing.

We can generalize risk-neutral pricing with the equation

$$D(t)S(t) = \mathrm{E}^{\mathbb{Q}}[D(T)S(T)|\mathcal{F}(t)] \tag{2.2.10}$$

where $0 \le t < T$. This equation implies that the discounted price process is a $\mathbb{Q}$-martingale. When $S(t)$ follows GBM, we can derive risk-neutral pricing through the discounted price process as follows.

We will compute $d[D(t)S(t)]$, but first define a product rule for Itô processes. Apply Itô's lemma on $f(X(t)) = X(t)^2$ to obtain

$$df = f'(X(t))\, dX_t + \frac{1}{2} f''(X(t))\, (dX_t)^2 = 2X(t)\, dX_t + (dX_t)^2. \tag{2.2.11}$$

In general, the product xy is equivalent to $\frac{1}{2}((x+y)^2 - x^2 - y^2)$. By linearity of the differential operator, we have for two arbitrary Itô processes $X(t)$ and $Y(t)$

$$d[X(t)Y(t)] = \frac{1}{2}\left(d[X(t) + Y(t)]^2 - dX_t^2 - dY_t^2\right). \tag{2.2.12}$$

Simplifying the above and using equation (2.2.11), we obtain *Itô's product rule.*

Theorem 2.2.6. *For two arbitrary Itô processes $X(t)$ and $X(t)$, it follows that*

$$d[X(t)Y(t)] = X(t)\, dY_t + Y(t)\, dX_t + d[X(t), Y(t)]$$

where $d[X(t), Y(t)] = dX_t\, dY_t$ is the quadratic covariation of $X(t)$ and $Y(t)$.

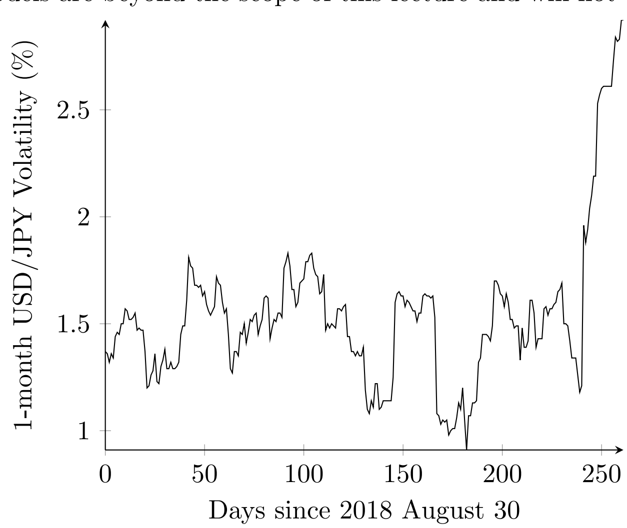
We can now apply Itô's product rule on $D(t)S(t)$ to obtain

$$\begin{aligned} d[D(t)S(t)] &= D(t)\, dS(t) + S(t)\, dD(t) + d[D(t), S(t)] && (2.2.13)\\ &= D(t)[\mu S(t)\, dt + \sigma S(t)\, dW_t] - rD(t)S(t)\, dt && (2.2.14)\\ &= (\mu - r)D(t)S(t) + \sigma D(t)S(t)\, dW_t && (2.2.15)\\ &= \sigma D(t)S(t)[\lambda\ dt + dW_t] && (2.2.16) \end{aligned}$$

where $\lambda = \frac{\mu - r}{\sigma}$ is the *market price of risk.* By Girsanov's theorem, there exists an equivalent measure $\mathbb{Q}$ such that $dW_t = dW_t^{\mathbb{Q}} - \lambda\, dt$. Then

$$d[D(t)S(t)] = \sigma D(t)S(t) dW_t^{\mathbb{Q}}$$

implies $D(t)S(t)$ is a $\mathbb{Q}$-martingale. This easy derivation of risk-neutral pricing may be why GBM is a popular choice in price models, but it should be noted a major pitfall is its assumption of constant volatility. However, stochastic volatility models are beyond the scope of this lecture and will not be discussed.



2.3 Currency Forwards and Futures

A *forward contract* is an agreement to buy or sell an asset at a specific future time and price. The price of a forward depends on several factors, such as the risk-free interest rate and the cost of carrying the asset. Assuming no carrying cost, we have the function

$$F(t) = S(t)e^{r(T-t)} \tag{2.3.1}$$

where $S(t)$ is the price of the asset at time t, r is the risk-free interest rate, and $(T-t)$ is the time till delivery. We require r to equal the risk-free interest rate since arbitrage would exist otherwise. When $S(t)$ follows geometric Brownian motion (GBM), there is an interesting relation between the spot and forward price.

Let $S(t)$ be the solution to the stochastic differential equation

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW_t. \tag{2.3.2}$$

Applying Itô's lemma on $F(t, S(t))$, we obtain

$$\begin{aligned} dF &= \left(\frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial S^2} \right) dt + \sigma S \frac{\partial F}{\partial S}\, dW_t && (2.3.3) \\ &= [\mu S e^{r(T-t)} - r S e^{r(T-t)}]\, dt + \sigma S e^{r(T-t)}\, dW_t. && (2.3.4) \end{aligned}$$

By equation (2.3.1) we have the differential equation

$$\frac{dF}{F} = (\mu - r)\, dt + \sigma\, dW_t. \tag{2.3.5}$$

The forward price process is also a GBM, but with a percentage drift of $\mu - r$.

We can also derive forward prices under a risk-neutral measure. Consider a zero-coupon bond paying 1 at time T. We will denote its value at time t as $B(t,T)$. It follows that

$$B(t,T) = \frac{1}{D(t)} \mathrm{E}^{\mathbb{Q}}[D(T)|\mathcal{F}(t)]. \tag{2.3.6}$$

The discounted payoff of a forward under $\mathbb{Q}$ equals

$$\frac{1}{D(t)} \mathrm{E}^{\mathbb{Q}}[D(T)(S(T) - K)|\mathcal{F}(t)] \tag{2.3.7}$$

where K is the delivery price of the forward. Since the above is a martingale, it follows that the above equals

$$\frac{1}{D(t)} \Big(\mathrm{E}^{\mathbb{Q}}[D(T)S(T)|\mathcal{F}(t)] - \mathrm{E}^{\mathbb{Q}}[D(T)K|\mathcal{F}(t)] \Big) = S(t) - KB(t,T) \tag{2.3.8}$$

Forwards have an initial value of 0, so we necessarily have

$$F(t) = K = \frac{S(t)}{B(t,T)} \tag{2.3.9}$$

as the price of the forward. One can also prove equation (2.3.9) is the forward price by contradiction—if it does not hold, then arbitrage exists. The proof is simple and therefore left as an exercise.

We will now apply this discussion to currencies. If we interpret currencies as assets with a known yield, then we can define a currency forward as

$$F(t) = S(t)e^{(r-r_f)(T-t)} \tag{2.3.10}$$

where r and r_f are the domestic and foreign risk-free interest rates, respectively. Notice that if we assume $S(t)$ is defined as a GBM (equation (2.3.2)), then $F(t)$ has the drift rate of $r - r_f$ percent. This helps explain why exchange rates depend on interest rates between countries.

Although currency forwards can hedge against foreign exchange risk, it might be difficult to find a counterparty, and forward contracts are generally non-cancellable. These shortcomings are fixed in the futures market. A *futures contract* is an exchange-traded derivative for buying or selling a standardized asset at a certain future time and price. Market participants can enter and exit futures positions. Overall, futures differ from forwards due to being cancellable, exchange-traded, and standardized.

Futures can be replicated by buying and selling forward contracts daily and settling cash one day after entering a contract. However, this requires negligible default risk and ample market liquidity. Even if these conditions are met, the extent at which the cash flows will hedge against price movements is uncertain.

We will denote the price at time t of a futures contract delivering the asset $S(t)$ at time T as $\mathrm{Fut}_S(t,T)$. Since the replication results in a contract worth zero, the futures price satisfies the equation

$$\frac{1}{D(t)}\mathrm{E}^{\mathbb{Q}}[D(T)(S(T) - \mathrm{Fut}_S(T,T))|\mathcal{F}(t)] = 0. \tag{2.3.11}$$

If the interest rate process in $D(t)$ is deterministic, then we immediately have

$$\mathrm{Fut}_S(t,T) = \mathrm{E}^{\mathbb{Q}}[S(T)|\mathcal{F}(t)] = S(t)e^{r(T-t)} \tag{2.3.12}$$

as the price of the futures contract. This looks identical to a forward, but there is a difference. Forwards use a zero-coupon bond as their numéraire whereas futures use a risk-free asset. For this reason, we distinguish forward prices with a *T-forward measure* defined by the Radon-Nikodým derivative

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{D(T)}{\mathrm{E}^{\mathbb{Q}}[D(T)]}.$$

In general, we have the following relation between forwards and futures.

$$\begin{aligned} F(t) &= \mathrm{E}^{\mathbb{Q}_T}[S(T)|\mathcal{F}(t)] = \mathrm{E}^{\mathbb{Q}_T}[\mathrm{Fut}_S(T,T)|\mathcal{F}(t)] \\ &= B(t,T)\mathrm{E}^{\mathbb{Q}}[D(T)\mathrm{Fut}_S(T,T)|\mathcal{F}(t)] = \mathrm{Fut}_S(t,T)e^{\sigma_B\sigma_{\mathrm{Fut}}\rho} \end{aligned}$$

The $e^{\sigma_B\sigma_{\mathrm{Fut}}\rho}$ term results from $B(t,T)$ and $\mathrm{Fut}_S(t,T)$ being correlated log-normal random variables. Deterministic interest rates imply $\sigma_B = 0$ and thus $F(t) = \mathrm{Fut}_S(t,T)$. Stochastic interest rates imply different prices.

2.4 Currency Swaps

Yet another way to manage foreign exchange risk is through currency swaps. Many types of currency swaps exist due to interest rate swaps and the flexibility of OTC derivatives. In this lecture, we will only discuss *fixed-for-fixed currency swaps*, where two parties lend each other currencies at a fixed interest rate.

The fixed-for-fixed currency swap begins with an agreed notional exchange.

$$\text{Company A} \underset{N_2}{\overset{N_1}{\rightleftharpoons}} \text{Company B}$$

Then both parties pay a fixed interest rate for an agreed period of time.

$$\text{Company A} \underset{r_2\%}{\overset{r_1\%}{\rightleftharpoons}} \text{Company B}$$

At the end of the swap, notional amounts are returned.

$$\text{Company A} \underset{N_2}{\overset{N_1}{\leftrightharpoons}} \text{Company B}$$

Table 2 is an example of such currency swap. Overall, Company A pays Japanese yen to receive U.S. dollars whereas Company B pays U.S. dollars to receive Japanese yen.

Year	A pays B	B pays A
0	$1,000,000	¥110,000,000
1	¥275,000	$22,500
2	¥275,000	$22,500
3	¥110,275,000	$1,022,500

Table 2: Hypothetical 3-year currency swap transactions

In order to value the above swap, compute the present value of each transaction. We will focus on Company B, who receives 0.25% annually in yen and pays 2.25% in dollars annually. Let the risk-free interest rate in the United States and Japan be 2% and −0.10%, respectively. Further suppose that the dollar-yen exchange rate is initially ¥110. Then the present value of the initial exchange is trivially zero. The forward dollar-yen exchange rate each year is, by equation (2.3.10), ¥107.71, ¥105.48, and ¥103.28. The next step is to convert the yen cash flows to dollars with the appropriate forward exchange rate. After doing so, discount all dollar cash flows by the risk-free interest rate in the United States. The sum of these discounted cash flows is the value of the swap. The value of this swap turns out to be $\$3,893.05$.

The purpose of these derivatives is usually to transform a form of debt into another form of debt. For instance, companies can borrow foreign currency with an interest rate is not achievable otherwise. Another usage is to reduce foreign exchange risk when handling certain cash flows in a foreign currency. Overall, currency swaps are highly flexible instruments used to transform various parameters in debt and the value of foreign cash flows.

2.5 Currency Options

While currency forwards, futures, and swaps are useful for hedging risk in certain future cash flows, they can not hedge risk in uncertain future cash flows. Options are able to do this kind of hedging. An *option* gives the right, but not the obligation, to buy or sell an asset at a certain price (called the *strike price*) before a certain *expiration date*. An option giving the right to buy is a *call option*, whereas an option giving the right to sell is a *put option*. Options are versatile derivatives, allowing many ways to transfer, mitigate, or assume risk.

One of the most fundamental questions in mathematical finance is options pricing. How do we "correctly" calculate the price of an option? We will begin with the famous *Black-Scholes model*, a continuous-time model published in 1973 by economists Myron Scholes and Robert C. Merton. The model makes the following assumptions.

1. There is a riskless asset earning the risk-free interest rate and a risky asset following geometric Brownian motion (GBM) that pays no dividends.
2. Assets can be bought and sold in arbitrary amounts.
3. There are no arbitrage opportunities or transaction costs.

Let $S(t)$ denote the value of the risky asset. We seek a formula for the value of a portfolio holding one option $V(t, S(t))$ at time t. Itô's lemma yields

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S}\, dW_t. \tag{2.5.1}$$

Suppose the portfolio also holds a certain quantity Δ of the risky asset $S(t)$. Then the value of our portfolio is

$$d(V+\Delta S) = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu\Delta S\right) dt + \left(\sigma S\frac{\partial V}{\partial S} + \sigma\Delta S\right) dW_t \tag{2.5.2}$$

which loses randomness when $\Delta = -\frac{\partial V}{\partial S}$. In other words, we have

$$d(V + \Delta S) = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt \tag{2.5.3}$$

and therefore a riskless portfolio! Since riskless portfolios earn the risk-free interest rate, it follows that

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt \tag{2.5.4}$$

resulting in the famous *Black-Scholes equation*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.5.5}$$

For call options, use the boundary condition $V(T, S(T)) = (S(T) - K)^+$, where K is the strike price and T is the time at expiration. As for put options, use the boundary condition $V(T, S(T)) = (K - S(T))^+$. While it is possible to solve this equation using risk-neutral measures and the Feynman-Kac formula, we will skip to the solutions for sake of brevity.

For a call and put option, we have

$$C(t, S(t)) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{2.5.6}$$

$$P(t, S(t)) = \Phi(-d_2)Ke^{-r(T-t)} - S(t)\Phi(-d_1) \tag{2.5.7}$$

where $\Phi(\cdot)$ is the cumulative distribution function for the standard normal distribution, and

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right)$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

where r is the risk-free interest rate, σ is the underlying asset's volatility, and the quantity $T - t$ is the time to expiration. Notice d_1 and d_2 are standard normal variables (this is easily verified with equation (2.2.5)). We assumed no dividends, but we will now weaken that assumption.

Assuming the underlying asset continuously pays dividends, we can define $S(T)$ under the risk-neutral measure $\mathbb{Q}$ as

$$S(T) = S(t)\exp\left(\left(r - q - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Z\right) \tag{2.5.8}$$

where $\tau = T-t$ is the time to expiration, $Z = \frac{W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)}{\sqrt{T-t}}$ is a standard normal variable, and q is a constant dividend rate. Notice $S(t)$ is $\mathcal{F}(t)$-measurable whereas $\exp\left(\left(r - q - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}Z\right)$ is independent of $\mathcal{F}(t)$. By equation (2.2.10), the price of a call option on $S(t)$ equals

$$C(t, S(t)) = \mathrm{E}^{\mathbb{Q}}[\exp(-r\tau)(S(T) - K)^+|\mathcal{F}(t)] \tag{2.5.9}$$

and after some expansion

$$\mathrm{E}^{\mathbb{Q}}\left[\exp(-r\tau)\left(S(t)\exp\left(\left(r - q - \frac{1}{2}\sigma^2\right)\tau - \sigma\sqrt{\tau}Z\right) - K\right)^+\right]. \tag{2.5.10}$$

Redefine d_1 as the quantity

$$\frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{S(t)}{K}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)\tau\right).$$

Then equation (2.5.10) is non-zero if and only if $Z < d_2$. It follows that

$$C(t, S(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} \left(S(t) e^{-\left(q + \frac{1}{2}\sigma^2\right)\tau - \sigma\sqrt{\tau} z} - K e^{-r\tau} \right) e^{-\frac{1}{2}z^2}\, dz \tag{2.5.11}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} S(t) e^{-\left(q + \frac{1}{2}\sigma^2\right)\tau - \sigma\sqrt{\tau} z - \frac{1}{2}z^2}\, dz - K e^{-r\tau} \Phi(d_2) \tag{2.5.12}$$

$$= \frac{S(t)}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-q\tau - \frac{1}{2}(z + \sigma\sqrt{\tau})^2}\, dz - K e^{-r\tau} \Phi(d_2) \tag{2.5.13}$$

$$= \frac{S(t) e^{-q\tau}}{\sqrt{2\pi}} \int_{-\infty}^{d_2 + \sigma\sqrt{\tau}} e^{-\frac{1}{2}z^2}\, dz - K e^{-r\tau} \Phi(d_2) \tag{2.5.14}$$

$$= S(t) e^{-q(T-t)} \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2) \tag{2.5.15}$$

is the price of the call option. Calculations similar to the above result in

$$P(t, S(t)) = K e^{-r(T-t)} \Phi(-d_2) - S(t) e^{-q(T-t)} \Phi(-d_1) \tag{2.5.16}$$

as the price of the put option.

If we treat a country's risk-free interest rate as their currency's "dividend rate," we can use equations (2.5.15) and (2.5.16) to price currency options. Let $r = r_d$ be the domestic risk-free interest rate and $q = r_f$ be the foreign risk-free interest rate. One interesting aspect of this pricing model is when the put and call prices are identical. In other words, when

$$C(t, S(t)) - P(t, S(t)) = 0.$$

Subtracting equation (2.5.18) from (2.5.17), we have

$$C(t, S(t)) - P(t, S(t)) = S(t) e^{-r_f(T-t)} - K e^{-r_d(T-t)} \tag{2.5.17}$$

which is referred to as *put-call parity*. This equation implies the prices of put and call options are equal when

$$K = S(t) e^{(r_d - r_f)(T-t)}$$

which is the forward exchange rate (equation (2.3.10))! Buying a call and writing a put with such K is identical to buying a forward. For instance, according to Table 3, the 90-day dollar-yen forward exchange rate was ¥107.50.

Put Price	Expiration	Strike Price	Call Price
¥1.21	2019 December 17	¥107.00	¥1.72
¥1.41	2019 December 17	¥107.50	¥1.41
¥1.65	2019 December 17	¥108.00	¥1.14

Table 3: Dollar-yen options as of 2019 September 17 at 16:40 JST

Put-call parity is not limited to currency options. In general, the relationship can be expressed as

$$C(t, S(t)) - P(t, S(t)) = D(t)[F(t) - K]$$

assuming a forward exists (or can be replicated) and the market is liquid (i.e. quick and easy to buy and sell the same asset). Note that these assumptions are the only ones we need. This relationship holds under assumptions that are even thinner than the Black-Scholes model. The following table is yet another example of put-call parity observed in real life.

Call Price	Expiration	Strike Price	Put Price
$725	2020 December 18	$7,900	$625
$696	2020 December 18	$7,950	$696
$669	2020 December 18	$8,000	$719

Table 4: Nasdaq-100 option prices as of 2019 September 16

So far we have covered *European-style options*, which do not allow exercise until expiration. However, there are also options allowing "early exercise." These are called *American-style options*. Since early exercise may be optimal and the exercise strategy of the option buyer is unknown, American-style options lack general analytical solutions. We can derive solutions by either strengthening our assumptions or approximating the solution through numerical methods. Such topics are beyond the scope of this lecture and will not be discussed. Instead we will turn our attention to *binary options*, which offer only two outcomes.

Binary options are broadly categorized into *cash-or-nothing options* and *asset-or-nothing options*. In both types of options, the buyer of the option is paid if and only if the underlying asset's price is above the option's strike price. Otherwise they earn nothing. Consider a cash-or-nothing call option paying one unit of cash. Under the risk-neutral measure, we have

$$C(t, S(t)) = \frac{1}{D(t)} \mathrm{E}^{\mathbb{Q}}[D(T)\mathbf{1}_{\{S(t)>K\}}(T)|\mathcal{F}(t)] \tag{2.5.18}$$

$$= e^{-r(T-t)}\Phi(d_2) \tag{2.5.19}$$

as the price of the call option. Similarly, the price of an asset-or-nothing call equals

$$C(t, S(t)) = S(t)e^{-q(T-t)}\Phi(d_1). \tag{2.5.20}$$

Equations (2.5.20) and (2.5.21) imply European-style call options are equivalent to a long asset-or-nothing option with a short cash-or-nothing option paying the strike price. These computations readily apply to currencies when defining r and q as before. Binary options are easily accessible but are considered as a form of gambling due to malicious brokerages advertising binary options as low-risk investments and profiting by taking the opposite side of their clients' trades. Consequently, they are outlawed in many countries. Even if they are not outlawed, the legal framework in countries such as Singapore makes binary options trading virtually impossible without untrustworthy offshore brokerages.

Needless to say, options are high-risk instruments not suitable for all investors. We will conclude this lecture with a discussion on call option risk management. Recall the formula for a European call option. Several variables influence the option's price, such as interest rates, time, and volatility. Each variable in equation (2.5.15) will change over time, so it is important to understand the sensitivity of an option's price to changes in these variables.

Computing partial derivatives of $C(t, S(t))$ yields the so-called "Greeks." For instance, the *delta* of an option is defined as

$$\frac{\partial C}{\partial S} = e^{-q(T-t)}\Phi(d_1)$$

which is interpreted as the extent an option price will change in response to movements in the underlying asset's price. Traders often reduce their delta to zero by the end of the day in order to hedge against *gap risk*—the risk that an asset will dramatically fall or rise in price overnight. A portfolio with a delta of zero is called a *delta-neutral* portfolio.

Traders also monitor the second-order partial derivative *gamma*, which is defined as

$$\frac{\partial^2 C}{\partial S^2} = \frac{e^{-q(T-t)}\varphi(d_1)}{S(t)\sigma\sqrt{T-t}}.$$

Gamma is important to monitor while delta-neutral because large values of gamma mean that the delta-neutral position holds for a narrow price range.

Since options are influenced by not only the underlying asset price, but also its volatility, traders monitor *vega*, defined as

$$\frac{\partial C}{\partial \sigma} = S(t)e^{-q(T-t)}\varphi(d_1)\sqrt{T-t}.$$

Vega is often used to fine tune volatility exposure. For instance, a short-vega portfolio profits from the underlying asset's price trading within a relatively narrow range, whereas a long-vega portfolio profits from the exact opposite. Many long and short-vega portfolios are delta-neutral, since they risk a price's range rather than its direction.

There are also other Greeks such as *theta* (the quantity $\frac{\partial C}{\partial t}$) and *rho* (the quantity $\frac{\partial C}{\partial r}$). Rho is seldom used since most options expire under three months and short-term interest rates do not significantly change over such period. As for theta, it reminds us that options contain an extrinsic value that vanishes as the expiration date approaches. Theta is almost always negative when buying options. Overall, the partial derivatives of option prices are central to options risk management, as they help people understand the nuances of options.

A Further Reading

1. John C. Hull, *Options, Futures, and Other Derivatives (10^{th} ed.)*, Pearson, 2017.

2. Steven E. Shreve, *Stochastic Calculus for Finance II: Continuous-time Models*, Springer-Verlag New York, 2004.

B Review of Probability Theory

Definition B.0.1. Let Ω be a non-empty set and $\mathcal{F}$ be a σ-algebra of Ω. Then a *probability measure* P is a mapping $P : \mathcal{F} \to [0, 1]$ such that $P(\Omega) = 1$ and

$$P\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} P(A_i).$$

Definition B.0.2. The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space.*

Definition B.0.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is an *$\mathcal{F}$-measurable* random variable if $X^{-1}(B) \in \mathcal{F}$ for every Borel set B of $\mathbb{R}$.

Definition B.0.4. The *expectation* of a random variable X is a probability-weighed average defined as

$$\mathrm{E}[X] = \int_\Omega X\, dP.$$

Definition B.0.5. A random variable is *integrable* if $\displaystyle\int_\Omega |X|\, dP < \infty$.

Definition B.0.6. Let $(\Omega, \mathcal{F}, P)$ be as before. Then two sets A, $B \in \mathcal{F}$ are *independent* if $P(A \cap B) = P(A)P(B)$.

Definition B.0.7. Two random variables X, Y are *independent* if for any two Borel sets A, B of $\mathbb{R}$ it follows that $X^{-1}(A)$ and $Y^{-1}(B)$ are independent.

Definition B.0.8. Let X be an integrable random variable on the probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. Then $\mathrm{E}[X|\mathcal{G}]$ is a $\mathcal{G}$-measurable random variable such that

$$\int_A \mathrm{E}[X|\mathcal{G}]\, dP = \int_A X\, dP$$

for all $A \in \mathcal{G}$.

Definition B.0.9. A sequence of σ-algebras $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n$ is called a *filtration.*

Definition B.0.10. A *stochastic process* $X : \Omega \times T \to S$ is a family of random variables indexed by T and defined on a common probability space. Stochastic processes are often denoted as $X(t)$, $X(t, \omega)$, or X_t, where $t \in T$ and $\omega \in \Omega$.

Definition B.0.11. A stochastic process X is a *martingale* if $\mathrm{E}[X_t|X_s] = \mathrm{E}[X_s]$ for all $s \leq t$.

Definition B.0.12. An *$\mathcal{F}_i$-adapted process* is a stochastic process that is measurable by some filtration $(\mathcal{F}_i)_{i\in I}$.