#lyceum Lectures

desu-cartes

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Contents

1 Introduction to Manifolds

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The goal for this short lecture will be to define a smooth manifold (whatever that is!!!!). We rapidly begin building up elementary topology and then throw the definition of manifold at the reader, and then we will elabourate on what it means to be smooth. This will serve as the basis for further lectures on differential forms and Lie groups.

Preamble

Euclidean space is the variety of coordinate space we deal most often with in our day to day life. We envision the space we live in as a three-dimensional Euclidean space where a point in space is represented as a triple, $(x, y, z) \in \mathbb{R}^3$. The key idea about Euclidean space is that it is 'flat, where there is no general curvature of the space as opposed to objects like the sphere or a donut. Euclidean space is generalized to any dimension, being represented by the set of n-tuples of real numbers, denoted $\mathbb{R}^n$.

The difficult thing about manifolds is that they are like pornography: difficult to define, but you know it when you see it. Our working definition will be that a manifold generalizes our notion of "Euclidean space" in that, though looking globally, we might not immediately recognize it as flat, if we zoom in on the manifold, it will resemble closely to Euclidean space up to "continuous deformation" such as stretching and moving (but no ripping!).

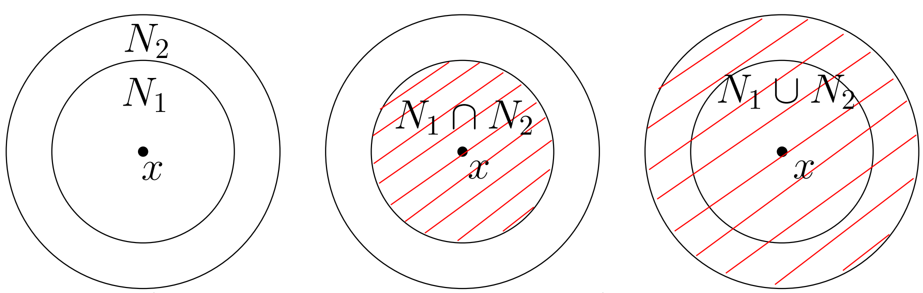
In order to discuss such continuous deformation, we need to develop a general enough concept of continuity. We do this via topology. Topology deals with all things stretching, shrinking, and growing. As long as it is a continuous deformation, topology has something to say. What we first will do is discuss the notion of "closeness" which will help us better approximate what it means to continuously deform a space into another.

1.1 Topology

Suppose we have a set X of points. We wish to discuss the notion of closeness between these points. If we have a metric—that is, a distance function—we can easily say what it means for points to be close together by discussing balls around points, specifying certain radii of the balls. This is not as general as we require for topology, as we would like to abandon the idea of a metric altogether. What we first do is we take our set X and specify a collection of "neighbourhoods" (which will be subsets of X) in our set which characterize when points are sufficiently close to one-another. In a metric space, these neighbourhoods often correspond with the aforementioned balls. Along with our English notion of "neighbourhood" we have some expectations:

(i) If we have two neighbourhoods, their union forms another neighbourhood.

(ii) if we have two neighbourhoods, their intersection forms another neighbourhood.



If N_1 and N_2 are both neighbourhoods of a point x, then their intersection $N_1 \cap N_2$ and union $N_1 \cup N_2$ are also both neighbourhoods of x.

These are reasonable expectations. We now formally define what it means by a topology on the set X, hoping to draw a parallel between our expectations of "neighbourhoods", and what we define to be neighbourhoods.

Definition 1 (Topology on X)**.** A *topology* on a set X is a collection of subsets J such that:

(i) $X, \emptyset \in J$;

(ii) Any arbitrary union of sets in J is again in J;

(iii) Any finite intersection of sets in J is again in J.

Together, we call (X, J) a *topological space*. Elements in J will be called *open sets* of X. A few peculiarities arise in this definition. Demanding that the whole set X is an open set is not too much to ask as it still agrees with preconceived notions of neighbourhoods, but we also demand the empty set to be open. The reason behind this is more of a corollary of formality. For now we can accept it as it is, but the reason the empty set is open is because when we find the induced "topology" from a metric on a set, this is the case. Another peculiarity is that in (ii) we suggest we can take any arbitrary union (that is, we can take infinite unions, or finite ones) of open sets and it will be again open, but only finite intersections of open sets are open. This peculiarity also finds itself as a result of the induced topology from a metric, but we will say a bit more on it later.

Example 1. Consider the real number line, $\mathbb{R}$. The topology on R is formed by taking the collection of open intervals of the form (a, b) and extending this set by adding all the arbitrary unions of sets of this form. That is, a set is open in $\mathbb{R}$ if and only if it is a union of open intervals.

Note that $\mathbb{R}$ is the union of all the open intervals, and the empty set is the union of no open intervals. It's not difficult to check the rest of the axioms of a topology are satisfied. For a second, assume we are allowed to call infinite intersections of sets open. Consider the family of open sets $\mathcal{F} := \{(-1/n, 1/n) \subset \mathbb{R} \mid n > 1\}$. Take the infinite intersection across this family of open sets. You get that $\bigcap \mathcal{F} = \{0\}$. This doesn't seem to respect what we called neighbourhoods in $\mathbb{R}$, nor does it seem to respect our preconceptions of neighbourhoods. This gives a glimpse into why we do not require infinite intersections of open sets to be open.

A definition I am not sure will come in handy:

Definition 2. A *closed set* in a topological space (X, J) is a subset $K \subseteq X$ such that its complement $K^C := \{x \in X \mid x \notin K\}$ is open.

Exercise 1. What are the closed sets in $\mathbb{R}$ according to our definition of its standard topology above?

Now we will look at the idea of "continuous" functions between topological spaces.

We won't take for granted that you know what the epsilon-delta definition of continuity is. But we can sketch the idea of continuity of a function $f\colon X \to Y$ between two topological spaces (we keep the topologies of these spaces implicit, referring to "open in X" and "open in Y") in terms of neighbourhoods as follows:

"The function f is continuous at $x \in X$ if any neighbourhood of x gets mapped to a neighbourhood of $f(x)$."

That is, if you stay close enough to x, you get mapped by f closely to $f(x)$. With some mathemagical trickery we make the following definition:

Definition 3 (Continuous function)**.** Let X and Y be topological spaces. A function $f\colon X \to Y$ is *continuous* if the pre-image,

$$f^{-1}[O] := \{x \in X \mid f(x) \in O\},$$

of all open sets O in Y is open in X.

When considering $\mathbb{R}^n$, this coincides with your usual definition of continuity you learn in analysis or calculus or whatever.

For those wondering about the mathemagical trickery, it is outlined in appendix A (assuming you get the idea of a "metric space", but nothing more).

Exercise 2. Show that the identity function $\mathrm{id}\colon X \to X$ on any topological space X defined by $\mathrm{id}(x) = x$, for all $x \in X$, is continuous.

Now the continuous deformation we wished to speak about in the beginning is not quite just "continuous" functions. Consider the function $g\colon [0, 2\pi) \to S^1$ defined by:

$$g(\theta) = (\cos\theta, \sin\theta),$$

where S^1 denotes the unit circle. The function is continuous, and also has the property of being bijective (meaning that it admits an inverse function, g^{-1}. The function however seems to fail at preserving shape, as it effectively deforms the interval $[0, 2\pi)$ (a line) into S^1 (a circle). That is, it glues one end of the interval to the other! This is not an example of "continuous deformation" since gluing doesn't result from the bending, shrinking, and growing movements that we have emphasized. So "continuous" is not enough!

The problem with g is that the inverse g^{-1} is discontinuous at the point $(1, 0)$, because every neighbourhood about $(1, 0)$ on S^1 is the union of two intervals:

$$[0, a) \cup (b, 2\pi),$$

which happens to be not be open when pulled back to S^1, given what is called the *subspace topology* inherited from $\mathbb{R}^2$ (the subspace topology calls sets open if there the set is a restriction of an open set in the larger topology, in this case $\mathbb{R}^2$).

To fix this sort of problem, we characterize continuous deformation using the following type of function:

Definition 4 (Homeomorphism)**.** A function $f\colon X \to Y$ between topological spaces is called a *homeomorphism* if it is a continuous bijection where its inverse f^{-1} is also continuous.

Effectively this allows us to continuous deform X into Y via f, and then undo it using f^{-1} in a continuous way (which was impossible with the example of g above). In the case that $f\colon X \to Y$ is a homeomorphism, we say that X is *homeomorphic* to Y.

Definition 5 (Locally Euclidean)**.** A topological space X is said to be *locally Euclidean* if for every $x \in X$ there is an open set U in X and a function $\varphi\colon U \to \mathbb{R}^n$ such that:

(i) $x \in U$;

(ii) φ is a homeomorphism onto its image.

The pair (U, φ) is called a *chart* on X, the collection of charts constituting an *atlas* on X.

Perhaps in more accessible language: a locally Euclidean space is one which has a neighbourhood around each point that resembles Euclidean space in that we can continuously deform the neighbourhood to look like it is flat!

1.2 Manifolds and smoothness

We begin with a definition:

Definition 6 (Hausdorff space)**.** A topological space X is *Hausdorff* if for any two distinct points $x, y \in X$, there exists two disjoint, open sets $U, V \subset X$ such that $x \in U$ and $y \in V$.

Effectively, the so-called Hausdorff property provides a sort of degree of separation for a topological space, giving any two distinct points disjoint open neighbourhoods. This is akin to Euclidean space, and it will play a subtle role in the study of manifolds.

A little thing to note is that we are going to define what it means to be a "manifold", but there are several views on what a manifold should be. Some authors choose to simply call a locally Euclidean space a topological manifold and that is it. However, the majority of authors like to include a few more properties (such as the Hausdorff property) which give the structure of a manifold nicer-behaved properties.

With this in mind, we apologize as now we are going to throw a large amount of relatively-unmotivated terminology into the mix, in hopes that we can just take them for granted and move forward to our goal.

Definition 7 (Basis for a topology)**.** Let X be a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that:

(i) For all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$.

(ii) If $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$, then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

We define a topology J on X generated by $\mathcal{B}$ by: $O \in J \iff$ each $x \in O$ has a $B \in \mathcal{B}$ such that $B \subseteq O$. So a basis for a topology gives us a simpler way of describing a topology. For example, we almost used a basis to describe the topology for $\mathbb{R}$. A basis for this topology is the set of all open intervals.

Definition 8 (Second-countable)**.** A topological space X is *second-countable* if it admits a countable basis for its topology.

Definition 9 (Topological manifold)**.** A *topological manifold* is a topological space M which is:

(i) Locally Euclidean (that is, it has an atlas);

(ii) Hausdorff;

(iii) Second-countable.

In the case each chart in the atlas of M maps into $\mathbb{R}^n$ for a fixed n, we say that M has *dimension* n, or that it is an n-manifold, for short.

Now a few examples to take for granted. Feel free to try your hand at proving that each is locally Euclidean:

1. Unsurprisingly, $\mathbb{R}^n$ is an n-manifold.
2. The sphere, denoted by S^2, is a 2-manifold.
3. The torus (or donut), denoted by T^2, is a 2-manifold.
4. The circle, S^1, is a 1-manifold.
5. The graph of a continuous function $f\colon U \to \mathbb{R}^n$ where $U \subseteq \mathbb{R}^m$ is an m-manifold. It has only one chart, being the projection on the first coordinate of the graph.
6. If M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold.

Now that we know what manifolds are, we are without a goal. So let's explore what we can do with a manifold. Since it is locally Euclidean, what can we do with Euclidean space that we might want to do in a more general scenario on a manifold? The first and only thing on the mind of any highschool or first-year student should be "calculus". What would it take to do calculus on manifolds? This is an excellent goal. So now we want to develop topological manifolds in order to do calculus on them. If O is an open set in $\mathbb{R}^n$, a function $g\colon O \to \mathbb{R}^n$ is called *smooth* if and only if g is infinitely differentiable. That is, $\frac{\partial^n g}{\partial x^n}$ exists for all n.

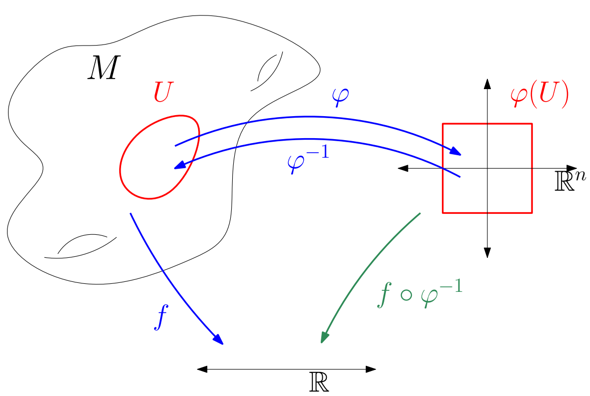
For most people, it is more comfortable discussing maps $O \to \mathbb{R}$ than into $\mathbb{R}^n$, so we will stick with that for now. If we have a map $f\colon M \to \mathbb{R}$ on a manifold M, what might it mean for it to be "smooth"?

Since the domain of f is M, we cannot naively differentiate our function since M is not necessarily $\mathbb{R}^n$—that is, it's not always flat. But what we do have is the next best thing!

We have for any point p in M a neighbourhood U around p which we can flatten into an open subset of $\mathbb{R}^n$ via a chart $\varphi\colon U \to \mathbb{R}^n$. So now we have two ingredients, f and φ, and we need to combine them to hopefully make something we can differentiate, and hence discuss the smoothness of. There is only one sensible way of doing this, and that is by composing them:

$$f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{R}$$

where we take advantage of the fact that φ has an inverse $\varphi^{-1}\colon \varphi(U) \to U$.



Note that this is sort of our situation, we have made a function between $\mathbb{R}^n$ and $\mathbb{R}$ based on f and φ and now we can differentiate this composition (the green function).

It now might be tempting to just define f as "smooth" at p if there exists such a φ such that this composition $f \circ \varphi^{-1}$ is smooth in the usual sense. There remains a problem however: it might depend on the chart, φ!

Suppose that $\psi\colon V \to \mathbb{R}^n$ is another chart such that $p \in V$. We could equally call f (not) smooth if $f \circ \psi^{-1}$ is (resp. not) smooth. Note that with a little algebra of functions we get:

$$f \circ \psi^{-1} = f \circ (\varphi^{-1} \circ \varphi) \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1}),$$

on $\varphi(U \cap V)$, which is not always smooth in the usual sense of the word.

We would perhaps need both parenthetical parts $f \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ to be smooth. It's not hard to imagine a case where you can choose the charts so that only one of these compositions is smooth. In this way, our topological manifolds are not quite suited for discussing smootheness—not yet at least!

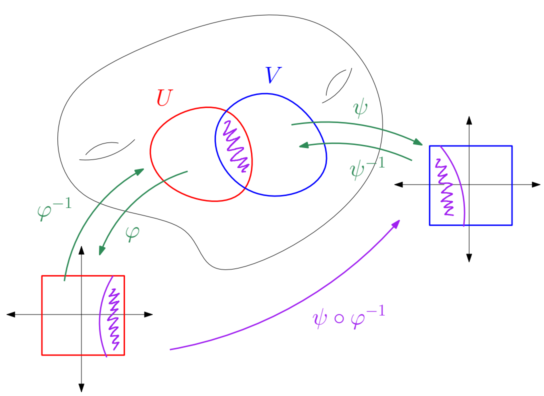
To make this working definition of smoothness not depend on charts, we can force the smoothness of $\varphi \circ \psi^{-1}$ using the following definition:

Definition 10 (Compatible charts)**.** Let M be a topological manifold. Two charts (U, φ) and (V, ψ) are said to be *compatible* if

$$\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$$

is a *diffeomorphism* (that is, it is bijective and both it and its inverse are smooth).

In this manner, if we have an atlas consisting of compatible charts, the smoothness of a function $f\colon M \to \mathbb{R}$ does not depend on the chart we choose. If it is smooth with respect to one chart, it is smooth with respect to all charts.



The purple function here is the one we require to be smooth in order for the charts to be compatible.

Definition 11 (Smooth atlas)**.** Let M be a topological manifold. An atlas $\mathcal{A}$ on M is called a *smooth atlas* if $\mathcal{A}$ is an atlas (the usual kind) such that if φ and ψ are charts in $\mathcal{A}$, then they are compatible.

Now when we can conclusively define that a function $f\colon M \to \mathbb{R}$ is *smooth* at $p \in M$ if there exists a chart φ containing p such that $f \circ \varphi^{-1}$ is smooth in the usual sense. That is, assuming M has a smooth atlas. Now we are not done quite yet. Consider a manifold M with two smooth atlases, $\mathcal{A}$ and $\mathcal{B} := \mathcal{A} \cup \{\psi\}$ for some chart ψ which is not in $\mathcal{A}$. Are these atlases inherently different from each other? If $f\colon M \to \mathbb{R}$ is a function on M, we can consider its smoothness at some $p \in M$ which has $\varphi \in \mathcal{A}$ and $\psi \in \mathcal{B}$ as charts around p:

$$f \circ \varphi^{-1} \text{ is smooth } \iff f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1}) \text{ is smooth}$$

, and hence adding a compatible chart ψ to $\mathcal{A}$ did not change which functions end up being smooth.

In the spirit of this, we say that two smooth atlases are *compatible* if their union is also a smooth atlas. The compatibility of smooth atlases gives us an equivalence relation where we can discuss the smooth structure of the manifold despite there being many smooth atlases that give the identical collection of differentiable functions.

Theorem 1. Let M be a manifold with a smooth atlas $\mathcal{A}$. Then there exists a unique *maximal smooth atlas* $\mathcal{M}$ containing $\mathcal{A}$ (with respect to inclusion).

This maximally smooth atlas gives us a canonical choice as to what smooth atlas to consider if any at all. And so now we have the following definition:

Definition 12 (Smooth manifold)**.** A *smooth manifold* is a topological manifold with a choice of maximal smooth atlas, called the *smooth structure.*

So now let's forget about "topological manifolds" Whenever we talk about "manifolds", the "smooth" part will be assumed and implicit.

N.B.: one can consider other differentiable structures for manifolds where it is only k-times differentiable rather than infinitely differentiable.

Example 2. See below:

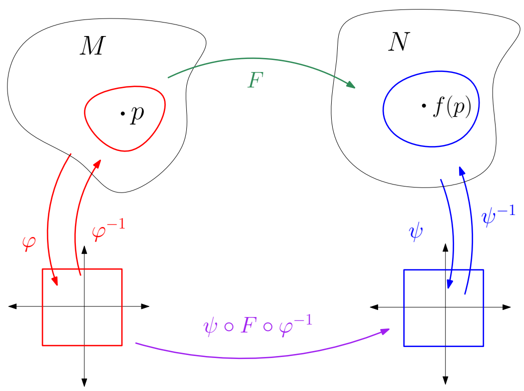
1. $\mathbb{R}^n$ is a smooth manifold.
2. S^2 is a smooth manifold.
3. Graphs of smooth functions are smooth manifolds.
4. The product of smooth manifolds is also a smooth manifold.
5. $\mathrm{GL}(n, \mathbb{R})$, the set of invertible $n \times n$ matrices with entries in $\mathbb{R}$ is also a smooth manifold.

We finish off this lecture by considering what it means for a map $F\colon M \to N$ between manifolds to be "smooth".

Definition 13 (Smooth map between manifolds)**.** A map $F\colon M \to N$ between manifolds is *smooth* at $p \in M$ if there exists a chart (U, φ) around p on M and a chart (V, ψ) around $F(p)$ on N such that $F(U) \subseteq V$ and:

$$\psi \circ F \circ \varphi^{-1}\colon \varphi(U) \to (\psi \circ F)(U),$$

is smooth in the usual sense.



This is the situation, and we require the purple map to be smooth in order for F to be smooth. This purple map is called the *transition map*. In a similar vein, if F is bijective, we say it is a *diffeomorphism* if both it and its inverse are smooth (i.e. the transition map is a diffeomorphism). Diffeomorphism is an equivalence relation on smooth structures.

A A bit about continuity

When we think about continuity intuitively, we often use the definition "if you can draw the graph of the function without lifting up your pen, then it is continuous. This of course has its limitations, though it still follows from the formal definition:

Definition 14 ($\varepsilon - \delta$ continuity)**.** A function $f : (X, d) \to (Y, d')$ between metric spaces is a continuous at a point $a \in X$ if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$ we have

$$d(a, x) < \delta \implies d'(f(a), f(x)) < \varepsilon.$$

In the case f is continuous at all $x \in X$, we call f itself continuous.

Intuitively, this definition says that for any small perturbation in Y (the ε), we can find a small perturbation in X (the δ) which f takes *inside* the perturbation in Y. That is, the perturbation in Y is controlled by that X.

Perhaps the word "perturbation" is not very accurate or well-thought out, but let us make another definition to ease our notation:

Definition 15 (Open ball)**.** Let (X, d) be a metric space and let $a \in X$. An open ball around a of radius $\delta > 0$ is the set

$$B(a; \delta) := \{x \in X : d(a, x) < \delta\}$$

Now, this definition should not be anything special, but we notice that the $d(a, x) < \delta$ occurs in our definition for continuity, so lets simply re-write the definition now with this new notation.

Definition 16 (Re-writing of continuity)**.** A function $f : (X, d) \to (Y, d')$ between metric spaces is continuous at a point $a \in X$ if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$ we have

$$x \in B(a; \delta) \implies f(x) \in B(f(a); \varepsilon),$$

or equivalently,

$$f[B(a; \delta)] \subseteq B(f(a); \varepsilon)$$

Now this definition conveys a fine subtlety that was not apparent when we were staring at the distance functions. Suppose f is continuous at $a \in X$. Then according to the above newly-notated definition we have that for any $\varepsilon > 0$ we have a $\delta > 0$ such that

$$B(a; \delta) \subseteq f^{-1}[B(f(a); \varepsilon)] := \{x \in X : f(x) \in B(f(a); \varepsilon)\},$$

that is, the pre-image of any ε-ball around $f(a)$ is a *neighbourhood* of a (by "neighbourhood" here, we mean a subset containing a and an open ball around a of some radius).

Suppose N' is a neighbourhood of $f(a)$ – that is, it contains an ε-ball around $f(a)$. Then we see that by virtue of the observation above, the pre-image of N' is a neighbourhood N of a. In conclusion, we have:

$$f \text{ continuous at } a \implies \text{ pre-images of neighbourhoods of } f(a) \text{ are neighbourhoods of } a.$$

Now we ask outselves, "what about the converse?" Well, let's suppose that pre-images of neighbourhoods of $f(a)$ are neighbourhoods of a:

Let $\varepsilon > 0$ and consider the ball $N' := B(f(a); \varepsilon)$ around $f(a)$. It is true that every open ball is a neighbourhood of each of its points (exercise!), so by our assumption, the pre-image $N := f^{-1}[B(f(a); \varepsilon)]$ is a neighbourhood of a. By definition of a neighbourhood, N contains a ball $B(a; \delta)$ around a, and it is clear then that $f(N) \subseteq N'$ and hence f is continuous at a. Summarily:

Theorem 2 (Neighbourhood definition)**.** A function $f : (X, d) \to (Y, d')$ is continuous at $a \in X$ if and only if pre-images of neighbourhoods of $f(a)$ are neighbourhoods of a.

Proof. The proof is embedded in the text above. □

Now we are not done here, though we might as well be. Consider the following generalization of an open ball:

Definition 17 (Open subset)**.** A subset $O \subseteq X$ of a metric space (X, d) is an open set if and only if it is a neighbourhood of each of its points.

For example, an open ball is an open set, or even the arbitrary union of open sets. Indeed:

Theorem 3. A subset $O \subset X$ of a metric space (X, d) is open if and only if it is a union of open balls.

Proof. Suppose O is open. Then for each $x \in O$ there exists an open ball $B(x; \delta_x) \subseteq O$. It follows that $O = \bigcup_{x \in O} B(x; \delta_x)$. On the other hand, suppose O is a union of open balls. Each $x \in O$ is in an open ball, just call it B, which is also an open set (by an exercise). Hence we have a $B(x; \delta) \subseteq B$. Hence O is a neighbourhood of each $x \in O$, and is open. □

With this in mind, we want to phrase continuity in terms of open sets, rather than open balls or neighbourhoods. So we finally consider the following characterization:

Theorem 4 (Open set definition)**.** A function $f : (X, d) \to (Y, d')$ is continuous if and only if the pre-images of open sets in Y are open in X.

Proof. Suppose f is continuous and let $O \subseteq Y$ be an open set in Y. For each $x \in f^{-1}[O]$ we have that O is a neighbourhood of $f(x)$. By continuity of f and Theorem 2, $f^{-1}[O]$ is a neighbourhood of x. Hence $f^{-1}[O]$ is open in X.

On the other hand, suppose the pre-image of each open set in Y is open in X. Let $a \in X$, and our goal will now be to show that f is continuous at a. Let N' be a neighbourhood of $f(a)$, then it contains an open ball B with $f(a) \in B \subseteq N'$. Because B is open, its pre-image $N := f^{-1}[B]$ is also open by assumption, and thus is a neighbourhood of a. Hence $f^{-1}[N']$ contains a neighbourhood of a Thus f is continuous at a. Because a was arbitrary, f is continuous. □

Now the conclusion here is that continuity only depends on the metric as much as it depends on the open sets. The metric gives rise to open balls which give rise to neighbourhoods which give rise to open sets, all of which we can define continuity by.

In the field of topology, we ignore the metric and open balls, but define the system of neighbourhoods or open sets of a space. We use observations about open sets in metric spaces to characterize open sets (the collection of which we call a topology), and it just so happens our characterization of open sets allow for more general spaces than metric spaces. That is to say, every metric space is a topological space, but not the other way around. In a topological space, Theorem 4 functions as the definition of continuity.