

A Sales Pitch for Combinatorics

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Recall from the basic counting review that binomial coefficients are defined as

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

and we showed a rather interesting result: that $\binom{2n}{n}$ is even for all $n > 0$. It turns out there are many equally interesting identities out there. We begin with a classical example.

Example 1.1. Suppose we form a 5-card poker hand from a standard 52-card deck¹. There are $\binom{52}{5}$ ways to do this. However, notice that every hand either contains the ace of spades or does not. If we form a hand by first including the ace of spades, we choose 4 cards from a deck of 51 cards. If we decide to not include the ace of spades, then we choose 5 cards from a deck of 51 cards. This yields $\binom{52}{5} = \binom{51}{4} + \binom{51}{5}$.

By replacing “deck” with “set,” “hand” with “subset,” and “ace of spades” with “distinguished element,” we essentially proved the following.

Theorem 1.2. (*Pascal’s Identity*) For all $0 \leq k \leq n$, we have $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

This identity allows us to compute binomial coefficients in an equally fast but significantly neater way than the absorption identity. It gives rise to what we call “Pascal’s triangle.”

$n = 0$				1						
$n = 1$				1	1					
$n = 2$				1	2	1				
$n = 3$				1	3	3	1			
$n = 4$				1	4	6	4	1		
$n = 5$				1	5	10	10	5	1	
$n = 6$				1	6	15	20	15	6	1

Example 1.3. The centermost numbers are precisely the values of $\binom{2n}{n}$, hence the name “central binomial coefficient.”

Example 1.4. Any number greater than 1 is the sum of the two numbers right above it; this is one “visualization” of Pascal’s identity.

Example 1.5. Summing the numbers in each row yields 1, 2, 4, 8, and so on. This is not a coincidence. Recall $[n]$ has 2^n subsets, so the sum of *all* k -element subsets of $[n]$ should be 2^n .

¹For readers unfamiliar with these cards, see https://en.wikipedia.org/wiki/Standard_52-card_deck.

Theorem 1.6. $\sum_{k=0}^n \binom{n}{k} = 2^n$.

Proof. The right-hand side counts the number of subsets of $[n]$. The left-hand side counts the number of k -element subsets separately and sums over all k , but these are all the subsets of $[n]$. \square

Theorem 1.7. $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$.

Proof. It suffices to show the number of even-size subsets of $[n]$ equals the number of odd-size subsets of $[n]$. Pick an element $x \in [n]$. The complement of an odd subset containing x is an even subset not containing x . On the other hand, the complement of an odd subset not containing x is an even set containing x . Summing the number of odd subsets thus yields the sum of even subsets. \square

We could also prove the above statement by invoking the binomial theorem, but that gives no intuition! Let us now move on to an example that we will use for the rest of this section.

Example 1.8. Suppose we are on a grid of integers, and we can move either north or east. How many paths² exist from, say, the origin $(0, 0)$ to the point $(2, 5)$? Drawing out such a path, we see that we must take exactly 2 steps east and 5 steps north for a total of 7 steps. But the positions of the east paths determine when we go north! For instance, $E = \{1, 3\}$ tells us to go east for the first and third steps, and north for the remaining steps. So the paths are really 2-element subsets of $[7]$, and we know there are $\binom{7}{2} = 21$ of those!

Lemma 1.9. There are exactly $\binom{n}{k}$ paths from $(0, 0)$ to $(k, n - k)$.

Theorem 1.10. (*Symmetry Identity*) $\binom{n}{k} = \binom{n}{n-k}$.

Proof. The left-hand side counts the number of paths from $(0, 0)$ to $(k, n - k)$, whereas the right-hand side counts the number of paths from $(0, 0)$ to $(n - k, k)$. Let $O = (0, 0)$, $P = (k, n - k)$, and $Q = (n - k, k)$. Then OP paths reflected along the line $y = x$ yield OQ paths, and vice versa. \square

We could also prove the symmetry identity by talking about k -element subsets of $[n]$, but the above “geometric” proof is nicer. In fact, let us prove Pascal’s identity using paths.

Proof of Theorem 1.2. Suppose we want to form a path from $(0, 0)$ to $(k, n - k)$. We can approach the last point in two directions: eastward from $(k - 1, n - k)$ or northward from $(k, n - k - 1)$. These sets of paths are inherently disjoint, so there are $\binom{n-1}{k-1} + \binom{n-1}{k}$ paths from $(0, 0)$ to $(k, n - k)$. \square

Let us add an interesting constraint to our paths. Let C_n enumerate paths from $(0, 0)$ to (n, n) that do not go above the main diagonal. The subtraction principle tells us C_n is equal to the number A_n of paths from $(0, 0)$ to (n, n) minus the number B_n of paths that go above the diagonal at some point. Clearly $A_n = \binom{2n}{n}$, but how do we count B_n ?

Lemma 1.11. There is a bijection between paths counted by B_n and paths from $(-1, 1)$ to (n, n) .

Proof. Let $O = (-1, 1)$ and P be a path counted by B_n . Then P intersects with line $y = x + 1$ at some point (why?). Let Q denote this point of intersection. Reflect the segment OQ along the line $y = x + 1$. This yields a path starting at $O' = (0, 0)$ where north and east steps are swapped until we reach the point Q . Since Q is fixed, we may recover P by reflecting $O'Q$ along $y = x + 1$. \square

Therefore

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n} - \frac{(2n)!}{(n+1)!(n-1)!} \cdot \frac{n}{n} = \binom{2n}{n} - \frac{n}{n+1} \frac{(2n)!}{n!^2} = \frac{1}{n+1} \binom{2n}{n}.$$

These numbers count so many things³ that we give them a special name: the *Catalan numbers*.

²Technically, these “paths” are north-east lattice paths, but for sake of brevity, we will just say “path.”

³Enough that there are entire books written on what they can count!

2 Graph Theory

So we have seen what enumerative combinatorics deals with, albeit in a very restricted scope due to lack of time and lack of diversity in examples. We will end this lecture by introducing yet another discrete structure and we will connect them with the previous discussion.

Definition 2.1. A *graph* is a pair $G = (V, E)$ where V is called a *vertex set* and E is called an *edge set*. Vertices are simply elements of V . Edges are 2-element subsets of V .

Definition 2.2. Given a graph G , we say a vertex u is *adjacent to* v if $uv \in E$. We also say E is *incident to* u (or v). The set of vertices adjacent to u is the *neighborhood* of u , denoted $N(u)$.

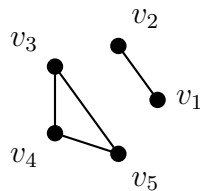
We assume V and E are finite and there are no loops or multiple edges. Note that vertices need not belong to an edge. In fact, V or E may be empty! When both V and E are empty, we have a *trivial graph*. When only E is empty, we have an *empty graph*.

Definition 2.3. The *order* of a graph is $|V|$, and the *size* of a graph is $|E|$. The number of edges incident to vertex v is the *degree* of v , written as $\deg(v)$.

Definition 2.4. A *walk* is a sequence of vertices v_1, \dots, v_n where $v_i v_{i+1} \in E$ for $1 \leq i \leq n - 1$. If edges are unique, then we have a *trail*. If both vertices and edges are unique, then we have a *path*.

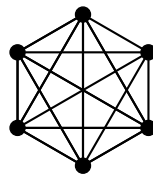
Definition 2.5. A *connected* graph is a graph where there is a path between any two vertices.

Example 2.6. Below is a graph of order 5 and size 4. The vertex v_4 , for instance, has degree 2. Notice there are no paths from v_1 to v_3 , and these vertices are not adjacent. Moreover, this graph is not connected, so this graph is *disconnected*. Lastly, $N(v_1) = \{v_2\}$.



Definition 2.7. A graph is *complete* if every vertex is mutually adjacent.

Example 2.8. The figure below is a complete graph on 6 vertices, which we denote as K_6 .



We will now show some basic results about graphs.

Theorem 2.9. Let G be a graph. Then $\sum_{v \in V} \deg(v) = 2|E|$.

Proof. Edges have two endpoints, so we count each edge twice when summing over each vertex. \square

Corollary 2.10. The number of vertices of odd degree in a graph must be even.

Theorem 2.11. Let u, v be vertices of a graph. Then a u - v walk contains a u - v path.

Proof. Given a walk $u = v_1v_2 \cdots v_n = v$, let k be the smallest integer satisfying $v_k = v_r$ for some $r > k$. Then construct the walk $u = v_1 \cdots v_kv_{r+1} \cdots v_n = v$ and repeat the process until no such k exists. As the walk is finite, this process eventually terminates. \square

We will now show some results reminiscent of those in the previous section.

Lemma 2.12. The maximum number of edges in a graph with n vertices is $\binom{n}{2}$.

Proof. First choose a vertex; there are exactly n of them. Each vertex can be connected to at most $n - 1$ other vertices. Since edges uv and vu are indistinguishable, we divide by 2 and hence the result holds. \square

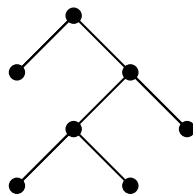
Theorem 2.13. A graph of order n with at least $\binom{n-1}{2} + 1$ edges is connected.

Proof. Suppose a graph is disconnected into two components with k vertices and $n - k$ vertices. By the above lemma, there are at most $\binom{k}{2}$ and $\binom{n-k}{2}$ edges, respectively. Since $\binom{k}{2} + \binom{n-k}{2} \leq \binom{n-1}{2}$ for $1 \leq k \leq n - 1$ (prove this), the result follows. \square

Since we have little time, we will see how graphs connect to Catalan numbers quickly, and, hopefully, discuss one beautiful connection between graph theory and group theory.

Definition 2.14. A *cycle* is a path with equal endpoints. A graph without cycles is *acyclic*. An acyclic connected graph is a *tree*. A *rooted tree* is a tree with a designated “topmost” vertex called a *root*. A *parent* of a vertex v is the adjacent vertex u lying on the path to the root. We also say v is a *child* of u . A *full binary tree* is a tree where every vertex has 0 or 2 children.

Example 2.15. How many full binary trees with $n + 1$ leaves are there? Exactly C_n . We will leave the reader to imagine (or search up) a bijection between the paths counted in the previous section and these kinds of trees.



Some graphs turn out to have virtually identical structure to other graphs in the sense that we can relabel vertices between two graphs without altering edges.

Definition 2.16. A *graph isomorphism* is a bijective function $f : G \rightarrow H$ where uv is an edge in G if and only if $f(u)f(v)$ is an edge in H . A *graph automorphism* is a graph isomorphism from a graph G to itself. The automorphism group of a graph is denoted as $\text{Aut}(G)$.

We now end this lecture with a quick example.

Theorem 2.17. $\text{Aut}(G) \cong S_n$ if G is an empty or complete graph on n vertices.

Proof. In an empty graph, the automorphisms are bijections from $[n]$ to itself, which is precisely S_n . As for a complete graph, notice that we may send any vertex anywhere without adding restrictions on the placement of the other $n - 1$ vertices since each vertex is mutually adjacent. \square

Let $G = (V, E)$ and K denote all 2-element subsets of V . The complement of G is defined as $\overline{G} = (V, K - E)$. Notice the complement of a complete graph is an empty graph. If you take a course in algebraic combinatorics, then you may see the more general result $\text{Aut}(G) \cong \text{Aut}(\overline{G})$.

Although this was not too enlightening, hopefully this entire lecture has shown you a different flavor of mathematics. Thank you for your time!